

UNIT-I

Introduction :-

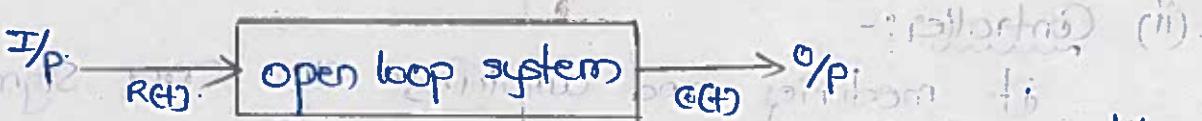
System :- when a no. of elements (or) Components are connected in a particular sequence to perform to specific output it's known as System.

Control System :- when a no. of elements (or) Components are connected in a particular sequence to perform to specific output, in a System, the output quantity is controlled by varying the input quantity is called control system.

Based on control action, the control systems are two types :-

- (i) Open loop Controlled System
- (ii) Closed loop Controlled System

(i) Open loop Controlled System :-



The Control system in which the output quantity has no effect on input quantity, is known as open loop control system.

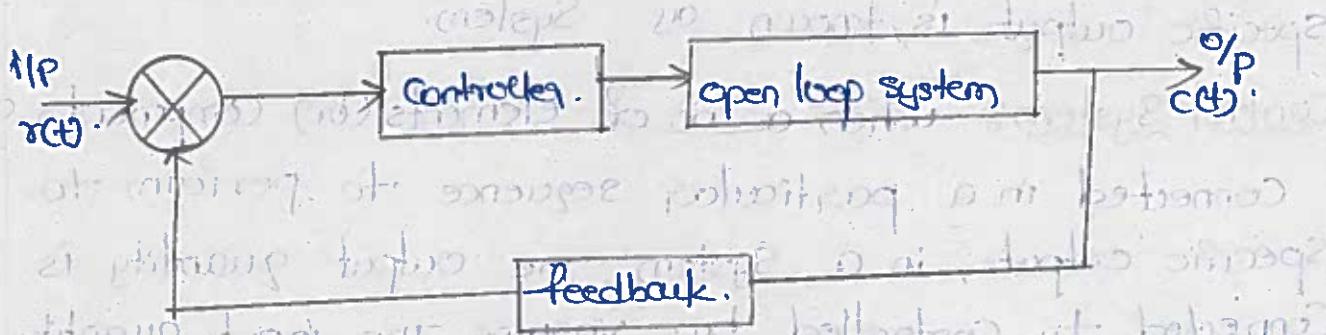
Any physical system, which doesn't automatically correct the variation in its output is called open loop control system.

Advantages of O.L.C.S :-

- * The open loop system is a simple and economical.
- * It is easier to construct.
- * Generally, the open loop systems are stable.

- * The open loop system are inaccurate and unreliable.
- * The changes in the output due to external disturbances are not corrected automatically.

(ii) Closed Loop Control System:-



The control system in which the output quantity has effect on the input quantity is known as the closed control system.

i) Error detector (or) Summing point (or) Comparator:-

It compares the difference b/w two input signals or known as error detector.

(ii) Controller:-

It modifies and amplifies the error signal.

(iii) Feedback:-
The feedback is a control action in which the O/P is sampled and a proportional signal is given to the I/P for automatic correction of any.

-Advantages of CLCS:-

* It is accurate and reliable.

* It is accurate even in the presence of non-linearity conditions.

* The sensitivity of the systems may be made small to make the system more stable.

* The closed loop system may be made to small to make the system more stable.

* The closed loop systems are less effected by the noise.

Disadvantages of CLS:-

- * It is complex and costly.
- * The feedback in this may lead to oscillatory response.
- * The feedback reduces overall gain of system.
- * Stability is a major problem in closed loop system and more care is needed to design a stable closed loop system.

Examples of Control System:-

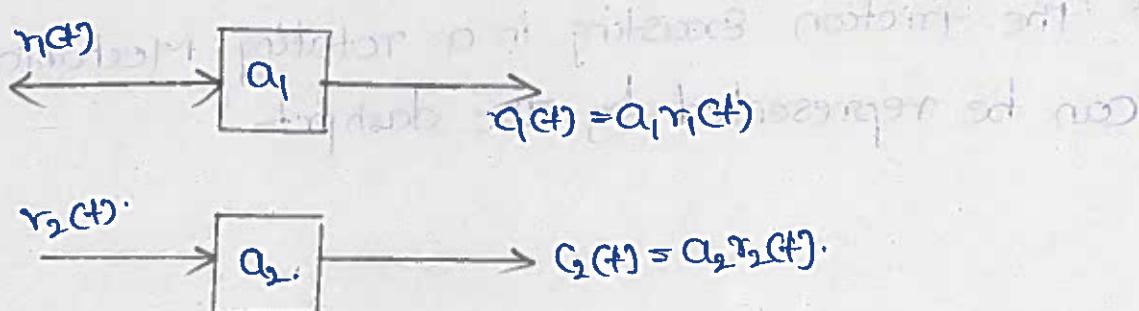
There are four types.

- (i) Temperature control system.
- (ii) Traffic control system.
- (iii) Numerical control system.
- (iv) Positional control system.

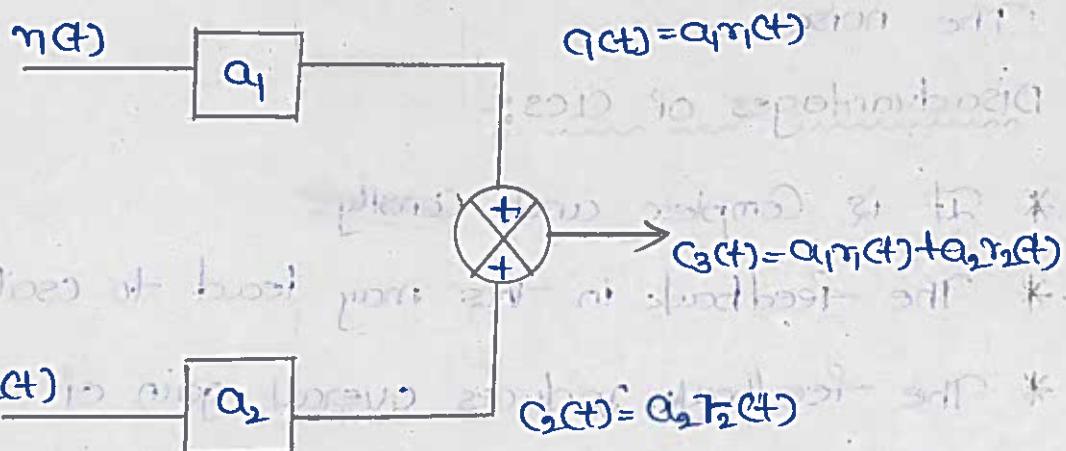
Mathematical Model of a Control System:-

The input and output relations of various physical components of a system are governed by differential equations. The Mathematical Model of a Control system consists of a set of differential equations.

The Mathematical Model of a control system is a linear if it obeys principle of Superposition and homogeneity.



The Mathematical Model is,



Transfer function :-

It is defined as the ratio of L.T of output to the Laplace transform of input with initial zero conditions.

Mechanical Systems :-

They are two types.

- i) Mechanical translation system
- ii) Mechanical Rotational system.
- iii) Mechanical translation system :-

It can be obtained by using 3 basic elements.

(i) Mass.

(ii) Dashpot

(iii) Spring.

- * The weight of Mechanical translation system is represented by the element mass and it is assumed to be connected at the center of the body.
- * The elastic deformation of the body can be represented by a spring.
- * The friction existing in a rotating Mechanical system can be represented by the dashpot.

Newton's second law :-

It states that sum of forces acting on a body is.

(or)

It states that sum of applied forces on a body are equal to sum of opposing forces on the body.

List of Symbols used in Mechanical Translation System :-

x = displacement = m

$v = \frac{dx}{dt}$ = Velocity = m/sec

$a = \frac{d^2x}{dt^2}$ = acceleration = m/sec²

f = applied force = newton's.

f_m = opposing force due to mass element = newton's.

f_b = opposing force due to dashpot element = newton's.

f_k = opposing force due to spring element = newton's.

M = Mass = kg.

B = dashpot = Newton.sec/m

k = Spring Newton = sec/m

Force balanced eqns. of idealised elements :-

i) Ideal Mass element :-



Consider an ideal mass element as shown in figure, which has negligible friction and spring let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body

let $f(t)$ = applied force.

f_m = opposing force due to mass element.

$$f_m \propto \frac{d^2x}{dt^2}$$

$$f_m \propto a.$$

$$\therefore f_m = M \cdot \frac{d^2x}{dt^2} \quad [\because a = \frac{d^2x}{dt^2}]$$

From the Newton's second law,

$$f = f_m$$

$$\therefore f = M \cdot \frac{d^2x}{dt^2}$$

2] Ideal friction element (or) dashpot element:-

$$x$$

$$F$$

$$B$$



Consider an ideal friction element as shown in figure, which has negligible mass and spring. Let a force be applied on it. The dashpot will offer an opposing force which is proportional to velocity of the body.

Let, f = Applied force.

f_b = opposing force due to dashpot element

$$f_b \propto v$$

$$f_b \propto \frac{dx}{dt}, \quad [\because v = \frac{dx}{dt}]$$

$$f_b = B \cdot \frac{dx}{dt}$$

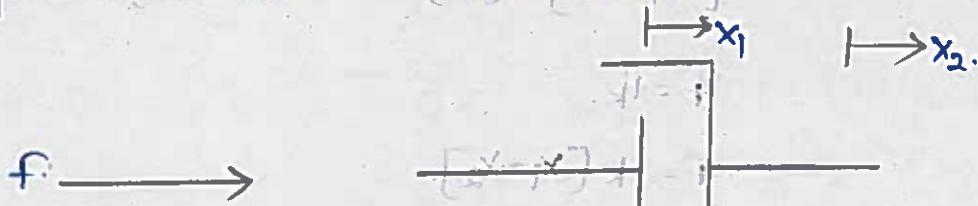
From the Newton's second law,

$$f = f_b.$$

$$\text{so } f_b = (B) \cdot \frac{dx}{dt}$$

$$\therefore F = B \frac{dx}{dt} \quad [or] \quad [dx/dt] \propto F$$

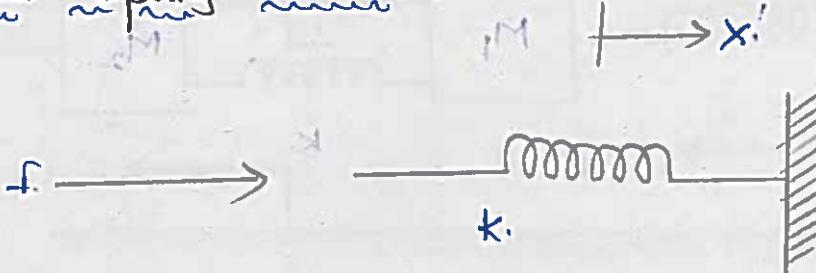
* Displacement at both ends as shown in figure.



$$f_b \propto \frac{d}{dt} [x_1 - x_2] \quad [or] \quad f_b \propto \frac{d}{dt} [x_2 - x_1]$$

$$f_b = B \frac{d}{dt} [x_1 - x_2] \quad [or] \quad f_b = B \frac{d}{dt} [x_2 - x_1]$$

3) Ideal spring element :-



Consider an ideal spring element as shown in figure, which has negligible mass and friction.

Let a force be applied on it, the spring will offer an opposing force which is proportional to displacement of the body.

Let f = Applied force

f_k = opposing force due to spring element

$f_k \propto x$

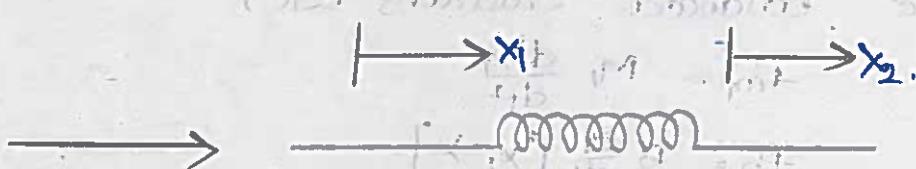
$$\therefore f_k = kx$$

from the Newton's Second law,

$$f = f_k$$

$$\therefore f = f_k$$

* Displacement at both ends as shown in figure.



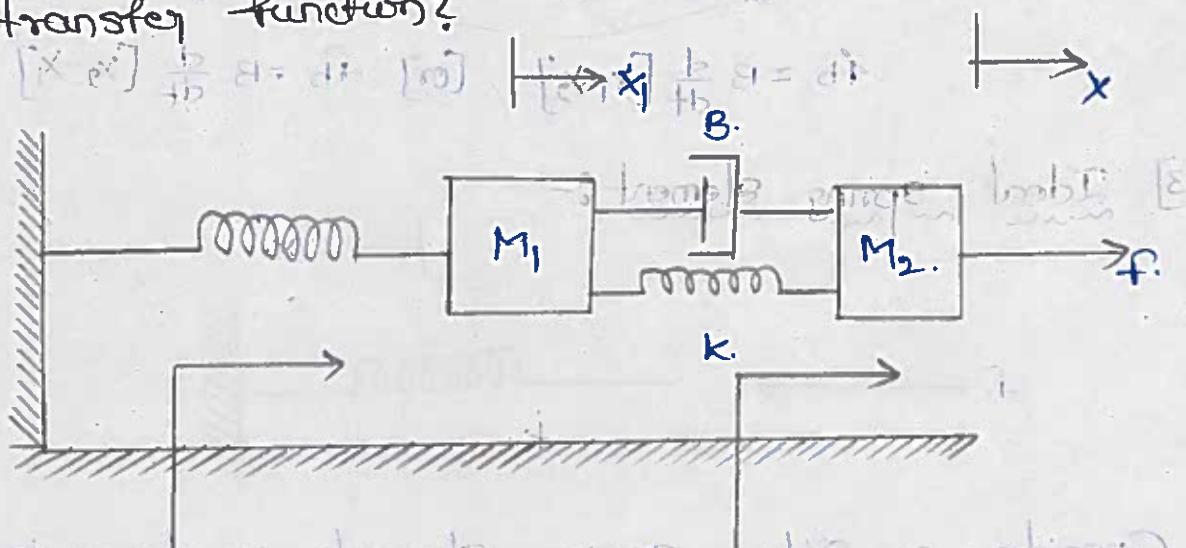
$$f_k \propto [x_1 - x_2] \quad [\text{or}] \quad f_k \propto [x_2 - x_1]$$

$$f_k = k[x_1 - x_2] \quad [\text{or}] \quad f_k = k[x_2 - x_1]$$

$$f = f_k.$$

$$f = k[x_1 - x_2].$$

- ① write the differential equations governing the mechanical system, shown in figure and determine the transfer function?



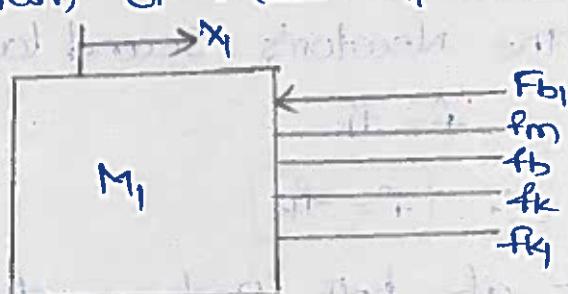
sol:- In the given system, applied $f(t)$ is the input, and displacement x is the output.

Laplace transform of input, i.e., $L[f(t)] = F(s)$

Laplace transform of output, i.e., $L[x] = X(s)$

$$\therefore \text{Required T.F, } T(s) = \frac{X(s)}{F(s)}$$

free body diagram of mass M_1 is shown in figure.



The force balanced equations are,

$$f_{m1} = M_1 \frac{d^2 x_1}{dt^2}$$

$$f_b = B \frac{dx}{dt} [x_1 - x]$$

$$f_k = k[x_1 - x].$$

$$(2) - f_{k_1} = k_1 x_1 \text{ is substituted}$$

$$(2) x [4 + Bs] [4 + Bs] - f_{b_1} = B_1 \frac{dx_1}{dt} [2Bs + 4 + 2B + B_1 s] \quad (2)$$

from the Newton's second law,

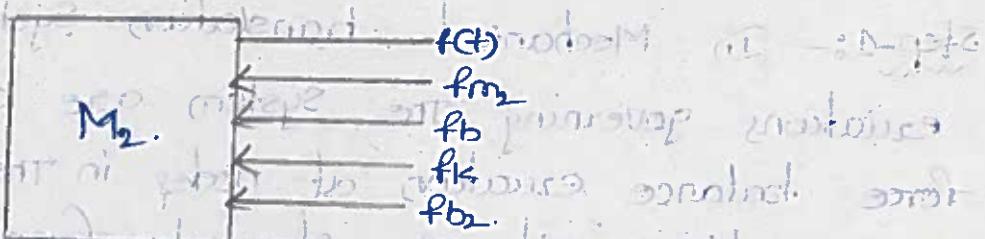
$$(2) \left[0 = f_{m_1} + f_B + f_k + f_{k_1} + f_{b_1} \right] \quad (2)$$

$$0 = M_1 \frac{d^2x_1}{dt^2} + B \frac{dx_1}{dt} [x_1 - x] + k [x_1 - x] + k_1 x_1 + B \frac{dx_1}{dt} \quad (2)$$

Applying L.T on b.s with zero initial conditions

$$0 = X_1(s) [M_1 s^2 + Bs + k + k_1 + B_1 s] - [Bs + k] X(s) \rightarrow (1)$$

free body diagram of mass M_2



The force balanced equations are;

$$f_{m_2} = M_2 \frac{d^2x}{dt^2}$$

$$f_b = B \cdot \frac{d}{dt} [x - x_1]$$

$$f_k = k [x - x_1]$$

$$f_{b_2} = B_2 \frac{dx}{dt}$$

from the Newton's second law,

$$f(t) = f_{m_2} + f_b + f_k + f_{b_2}$$

$$f(t) = M_2 \frac{d^2x}{dt^2} + B \frac{d}{dt} [x - x_1] + k [x - x_1] + B_2 \frac{dx}{dt}$$

By taking L.T on b.s with zero initial condition

$$F(s) = X(s) [M_2 s^2 + Bs + k + B_2 s] - X_1(s) [Bs + k] \rightarrow (2)$$

from equation (1)

$$X(s) = \frac{[Bs + k] X(s)}{[M_1 s^2 + Bs + k + k_1 + B_1 s]} \quad (3)$$

Substitute eqn -③ in eqn -②

$$F(s) = [M_2 s^2 + B s + k + B_2 s] x(s) - \frac{[B s + k][B s + k] x(s)}{M_1 s^2 + B s + k + k_1 + B_1 s}$$

$$F(s) = \left[[M_2 s^2 + B s + k + B_2 s] - \frac{[B s + k]^2}{M_1 s^2 + B s + k + k_1 + B_1 s} \right] x(s)$$

$$\therefore \frac{x(s)}{F(s)} = \frac{[M_1 s^2 + B s + k + k_1 + B_1 s]}{[M_1 s^2 + B s + k + k_1 + B_1 s][M_2 s^2 + B s + k + B_2 s] - [B s + k]^2}$$

⇒ Procedure for obtain the transfer function of Mechanical translation system :-

Step-1:- In Mechanical translation system, the differential equations governing the system are obtain by writing force balance equation at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass element in the system. In some case the nodes may be without mass element.

Step-2:- The linear displacement of masses are assumed as x_1, x_2, x_3, \dots etc. and assign a displacement to each mass. The first derivative of displacement is velocity, and second derivative of displacement is acceleration.

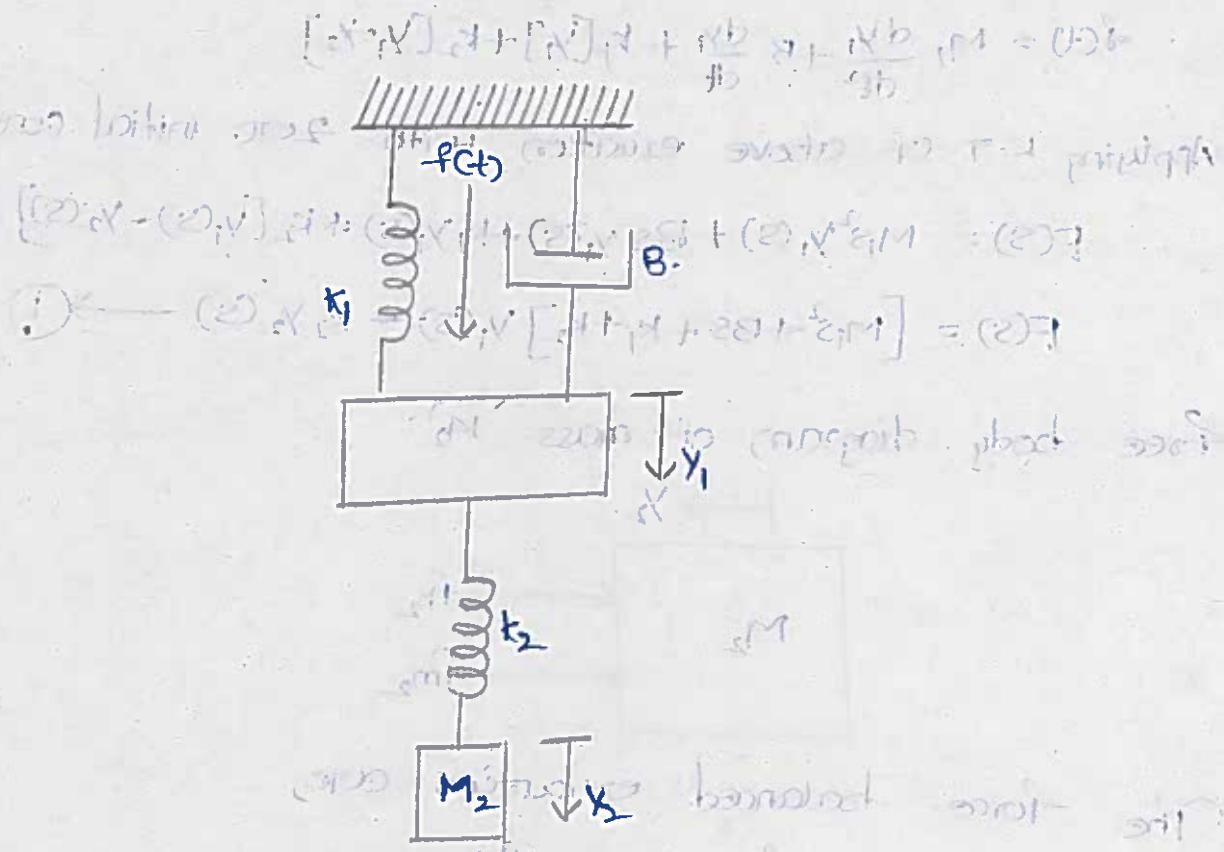
Step-3:- Draw the free body diagram of the system. The free body diagram is obtained by drawing each mass separately and then marking all the forces acting on that mass. Always the opposing force acting in a direction on that mass opposite to applied force; the mass has to move in the direction of the applied force.

Step-4:- for each freebody diagram, write one differential equation by equating the sum of applied forces for the sum of opposing forces.

Step-5:-

Take the laplace transform of differential equations to convert them to algebraic eqns. Then rearrange the s-domain equations to eliminate the unwanted variables, and obtain the ratio b/w output variable and input variable. This ratio is. The transfer function of the system.

- ② Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system shown in figure?



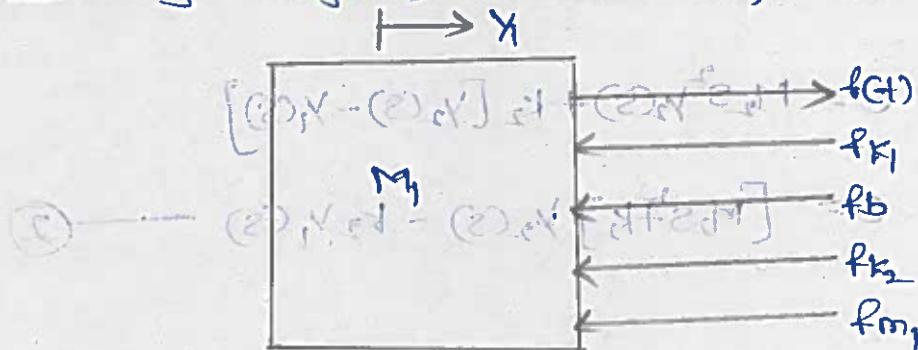
Sol:- In. The given system, applied force $f(t)$ is the input and -lement 'y' is the output.

$$\text{let L.T. of input, } L[f(t)] = F(s)$$

$$\text{L.T. of output, } L[y_2] = Y_2(s),$$

$$\therefore \text{Required T.F., } T(s) = \frac{Y_2(s)}{F(s)}$$

Free body diagram of mass 'm',



The force balanced equation's are,

$$f_{m_1} = M_1 \frac{d^2 y_1}{dt^2}$$

$$f_b = \frac{d}{dt} [y_1]$$

$$f_k = k_1 [y_1]$$

$$f_{k_2} = k_2 [y_1 - y_2]$$

from the Newton's 2nd law,

$$f(t) = f_{m_1} + f_b + f_k + f_{k_2}$$

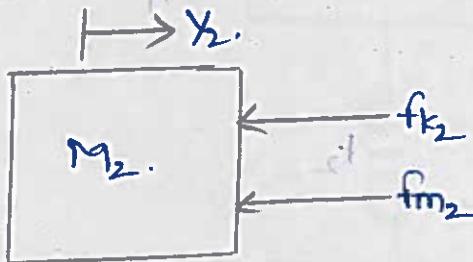
$$f(t) = M_1 \frac{d^2 y_1}{dt^2} + B \frac{dy_1}{dt} + k_1 [y_1] + k_2 [y_1 - y_2]$$

Applying L.T of above equation with zero initial condition

$$F(s) = M_1 s^2 y_1(s) + B s y_1(s) + k_1 y_1(s) + k_2 [y_1(s) - y_2(s)]$$

$$F(s) = [M_1 s^2 + B s + k_1 + k_2] y_1(s) - k_2 y_2(s) \rightarrow ①$$

free body diagram of mass M_2



The force balanced equation are,

$$f_{m_2} = M_2 \frac{d^2 y_2}{dt^2}$$

$$f_k = k_2 [y_2 - y_1]$$

from the Newton's second law,

$$0 = f_{m_2} + f_k : \text{zero initial condition}$$

$$0 = M_2 \frac{d^2 y_2}{dt^2} + k_2 [y_2 - y_1]$$

Applying L.T of above equation with zero initial condition

$$0 = M_2 s^2 y_2(s) + k_2 [y_2(s) - y_1(s)]$$

$$0 = [M_2 s^2 + k_2] y_2(s) - k_2 y_1(s) \rightarrow ②$$

(4)

From Eqn - ②

$$k_2 y_1(s) = [M_2 s^2 + k_2] y_2(s)$$

$$y_1(s) = \frac{[M_2 s^2 + k_2] y_2(s)}{k_2} \rightarrow ③$$

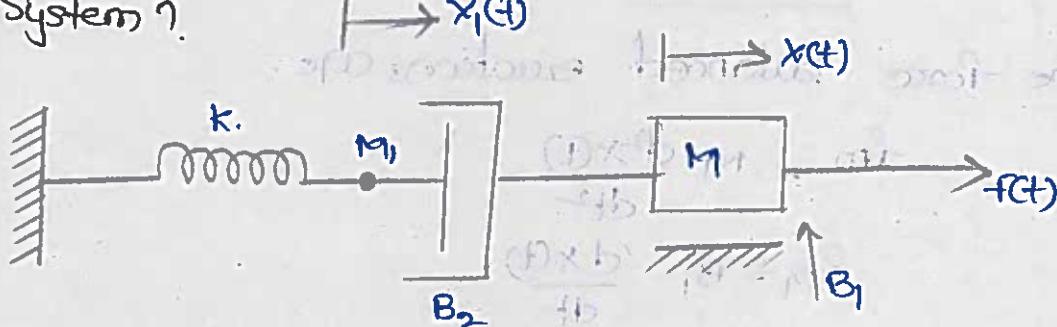
Substitute equation - ③ in Eqn - ①

$$F(s) = [M_1 s^2 + B_1 s + k_1 + k_2] \frac{[M_2 s^2 + k_2] y_2(s)}{k_2} - k_2 y_2(s)$$

$$F(s) = \left[\frac{[M_1 s^2 + B_1 s + k_1 + k_2] [M_2 s^2 + k_2]}{k_2} - k_2 \right] y_2(s)$$

$$\frac{y_2(s)}{F(s)} = \left[\frac{k_2}{[M_1 s^2 + B_1 s + k_1 + k_2] [M_2 s^2 + k_2] - k_2^2} \right]$$

- Q) write the Eqn of Motion in s-domain for the system shown in figure. Determine the transfer function of the system?



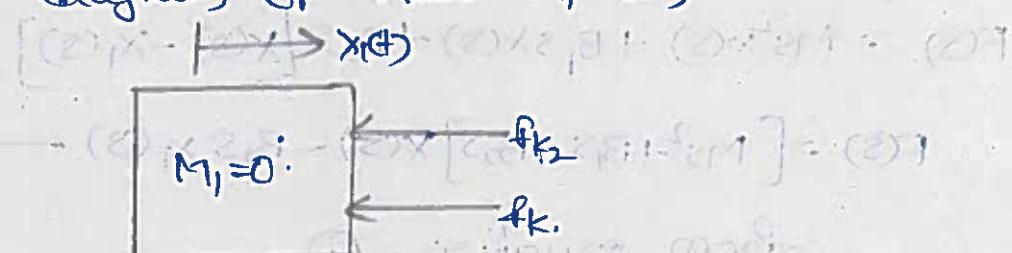
Sol:- In the given system, applied. $f(t)$ is the input and displacement ' $x(t)$ ' is the output

$$L.T. \text{ of input, } L[f(t)] = F(s)$$

$$L.T. \text{ of output, } L[x(t)] = X(s)$$

$$\therefore \text{Required T.F, } T(s) = \frac{X(s)}{F(s)}$$

free body diagram of mass ' M_1 ' is,



The force balance equations are,

$$f_{b_2} = B_2 \cdot \frac{d}{dt} [x_1(t) - x(t)]$$

$$f_k = k x_1(t)$$

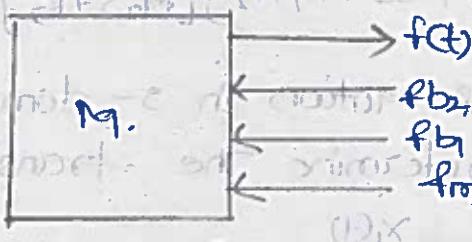
from Newton's second law:-

$$0 = f_{b_2} + f_k \\ = B_2 \cdot \frac{d[x_1(t) - x(t)]}{dt} + kx_1(t) \\ = B_2 \cdot \frac{d[x_1(t) - x(t)]}{dt} + kx_1(t)$$

- Apply L.T. on b.s.

$$0 = B_2 s [x_1(s) - x(s)] + kx_1(s) \\ 0' = [B_2 s + k] x_1(s) - B_2 s x(s) \rightarrow ①$$

The free body diagram of mass 'M' is,



The force balanced equations are.

$$f_m = M \frac{d^2 x(t)}{dt^2}$$

$$f_{b_1} = B_1 \frac{dx(t)}{dt}$$

$$f_{b_2} = B_2 \frac{d[x(t) - x_1(t)]}{dt}$$

from the Newton's second law,

$$f(t) = f_m + f_{b_1} + f_{b_2}$$

$$f(t) = M \frac{d^2 x}{dt^2} + B_1 \frac{dx}{dt} + B_2 \frac{d[x - x_1]}{dt}$$

- Applying L.T. on b.s.

$$F(s) = M s^2 x(s) + B_1 s x(s) + B_2 s [x(s) - x_1(s)]$$

$$F(s) = [M s^2 + B_1 s + B_2 s] x(s) - B_2 s x_1(s) \rightarrow ②$$

from equation - ①

$$[B_2 s + k] x_1(s) = B_2 s x(s)$$

$$X_1(S) = \frac{B_2 S X(S)}{B_2 S + K} \rightarrow ③$$

Sub $e_2^n - ③$ in $e_2^n - ②$

$$F(S) = [M S^2 + B_1(S) + B_2 S] X(S) - B_2 S \left[\frac{B_2 S X(S)}{B_2 S + K} \right]$$

$$= [M S^2 + B_1 S + B_2 S] - \frac{B_2 S [B_2 S]}{B_2 S + K} X(S)$$

$$\frac{F(S)}{X(S)} = \frac{[M S^2 + B_1(S) + B_2(S)]}{B_2 S + K} - \frac{[B_2 S]^2}{[B_2 S + K]}$$

$$\therefore \frac{X(S)}{F(S)} = \frac{B_2 S + K}{[M S^2 + B_1(S) + B_2(S)][B_2 S + K] - (B_2 S)^2}$$

2) Mechanical Rotational Systems:-

The mechanical rotational system can be obtained by using 3 basic elements.

- 1) Moment of inertia of mass.
- 2) Rotational friction [dashpot with R.F.]
- 3) Spring.

⇒ List of symbols used in Mechanical rotational system :-

θ = Angular displacement = radians.

$v = \frac{d\theta}{dt}$ = Angular Velocity = radians/sec

$a = \frac{d^2\theta}{dt^2}$ = Angular acceleration = rad/sec²

T = Applied torque

T_m = Opposing torque due to moment of inertia of mass.

T_b = Opposing torque due to dashpot

T_k = Opposing torque due to spring element.

J = Moment of Inertia of Mass.

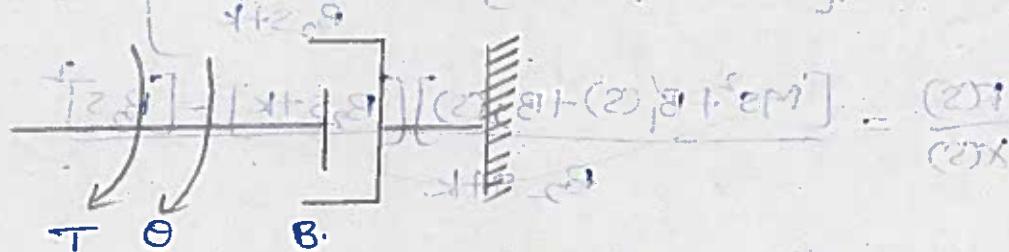
B = Rotational friction (or) dashpot.

$k = \text{Spring}$.

→ Torque balance equations of idealised elements:-

i) dashpot with rotational friction:-

Consider an ideal dashpot is shown in below figure.



which has negligible moment of inertia and spring let a torque applied on it. The dashpot will offer an opposing torque, which is proportional to angular velocity of the body

let, $T = \text{applied torque}$ and $\theta = \text{angle of rotation}$

$T_b = \text{opposing torque due to dashpot}$

$$T_b \propto \frac{d\theta}{dt} \quad (\text{from Newton's second law})$$

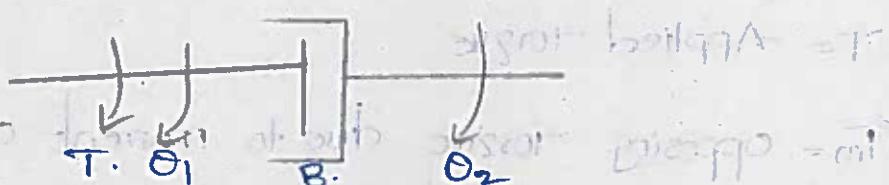
$$T_b = B \cdot \frac{d\theta}{dt} \quad (\text{from Newton's second law})$$

from Newton's second law

$$T = T_b \quad (\text{from Newton's second law})$$

$$T = B \cdot \frac{d\theta}{dt} \quad (\text{from Newton's second law})$$

Displacement at both ends,



$$T_b \propto \frac{d}{dt} [\theta_1 - \theta_2] \quad (\text{from Newton's second law})$$

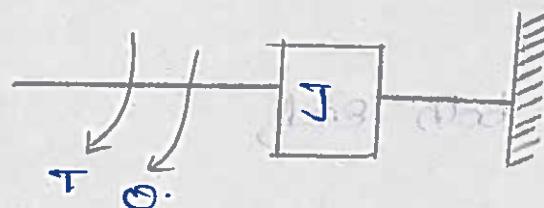
$$T_b = B \cdot \frac{d}{dt} [\theta_1 - \theta_2] \quad (\text{from Newton's second law})$$

from Newton's second law

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2] Moment of Inertia of Mass :-

Consider an ideal moment of inertia of mass as shown in below figure.



which has negligible rotational friction and spring. let a torque be applied on it. The moment of inertia of mass will oppose the torque, which is proportional to angular acceleration of the body.

let T = applied torque.

T_J = opposing torque due to moment of inertia of torque

$$T_J \propto \frac{d\theta}{dt}$$

$$T_J = J \frac{d^2\theta}{dt^2}$$

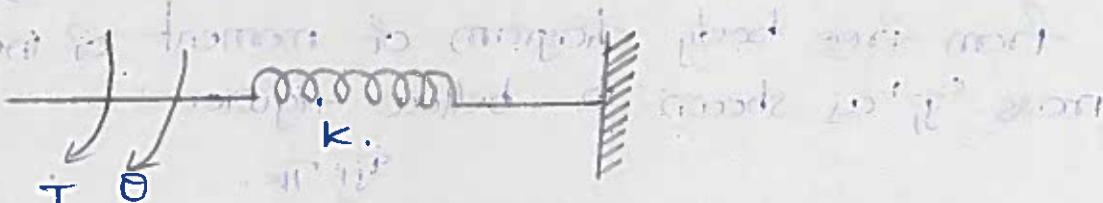
from Newton's second law,

$$T = T_J$$

$$T = J \frac{d^2\theta}{dt^2}$$

3) Spring:-

Consider, an ideal spring element is as shown in below figure.



which has negligible moment of inertia of mass and dashpot of rotational friction. let a torque be applied on it. The spring will offer an opposing torque, which is proportional to angular displacement of the body.

let, T = applied torque

T_k = opposing torque due to spring element

$$T_K \propto \theta$$

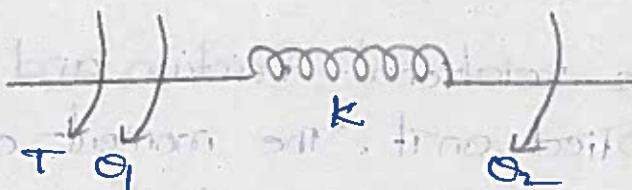
$$T_K = k\theta$$

Newton's second law;

$$T = T_K$$

$$T = k\theta$$

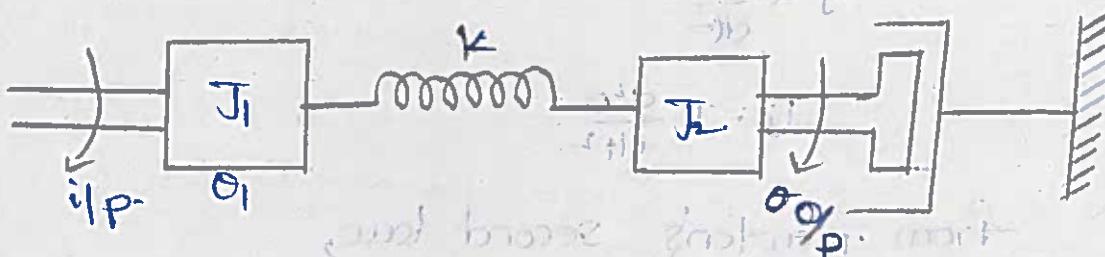
displacement at both ends,



$$T_K \propto (\theta_1 - \theta_2)$$

$$T_K = k[\theta_1 - \theta_2]$$

- ① write the differential eqns. governing the mechanical rotational system shown in figure. obtain the transfer function of the system?



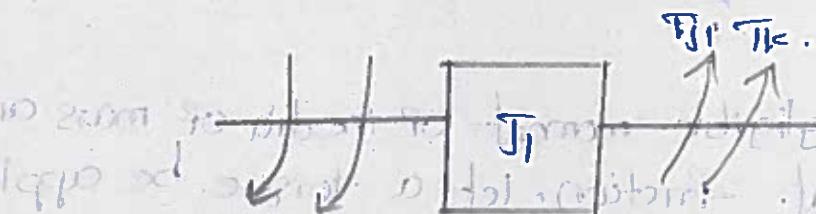
Sol:- In the given system, the applied torque 'T' is the input and angular displacement ' θ ' is the output.

$$\text{L.T of input, i.e., } L[T] = T[s]$$

$$\text{L.T of output, i.e., } L[\theta] = \theta(s)$$

$$\therefore \text{Required T.F, } T(s) = \frac{\theta(s)}{T(s)}$$

from free body diagram of moment of inertia of mass ' J_1 ' as shown in below figure.



The torque balanced equations are

$$T_f = J_1 \frac{d^2\theta}{dt^2}$$

$$T_k = k[\theta_i - \theta]$$

→ Newton's second law

(10)

from Newton's second law,

$$T = T_b + T_k.$$

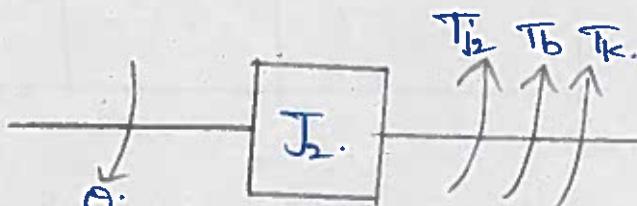
$$T = J_1 \frac{d^2\theta}{dt^2} + k[\theta_i - \theta].$$

Apply L.T of above equation with $\theta(0) = 0$ initial condition

$$T(s) = J_1 s^2 \theta_i(s) + k[\theta_i(s) - \theta(s)].$$

$$T(s) = [J_1 s^2 + k] \theta(s) - k \theta(s) \rightarrow (1)$$

free body diagram of moment of inertia of mass $J_2 \omega$ shown in below figure.



The torque balance equations are,

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} [0 - \theta]$$

$$T_k = k[\theta_i - \theta]$$

$$T_b = B \cdot \frac{d\theta}{dt}$$

from Newton's second law,

$$0 = T_{j2} + T_b + T_k.$$

$$= J_2 \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} + k[\theta_i - \theta]$$

$$\rightarrow J_2 s^2 \theta(s) + B s \theta(s) + k[\theta_i(s) - \theta(s)]$$

$$0 = [J_2 s^2 + B s + k] \theta(s) - k[\theta_i(s)]$$

$$\theta(s) = \frac{[J_2 s^2 + B s + k] \theta_i(s)}{k}$$

from (1)

$$T(s) = \underline{[J_1 s^2 + k][J_2 s^2 + B s + k] \theta(s) - k(\theta(s))}$$

$$\frac{\theta(s)}{T(s)} = \frac{k}{[J_1 s^2 + k][J_2 s^2 + B s + k] - k}$$

→ Electrical systems :-

The models of Electrical systems can be obtained by using 3 basic elements. They are,

1) Resistor

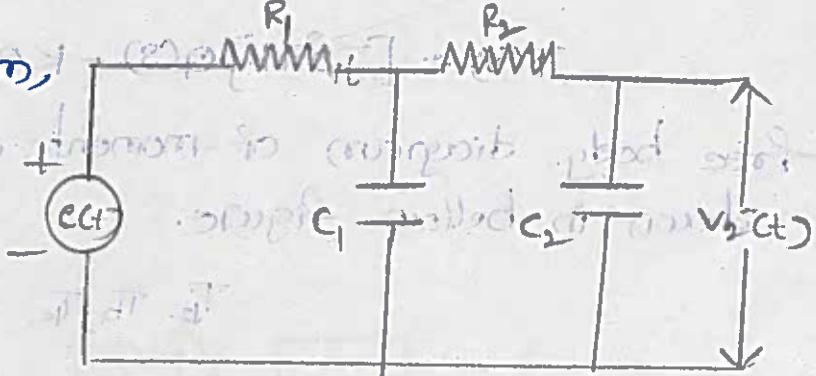
2) Inductor

3) Capacitor

Q1) Obtain the transfer function of electrical network shown in figure?

Sol:- In the given system,

The input is $E(t)$.
is $v_2(t)$



$$L.T \text{ of } i_p, L[E(t)] = E(s)$$

$$L.T \text{ of } o_p, L[v_2(t)] = V_2(s)$$

$$\therefore \text{Required T.F., } T(s) = \frac{V_2(s)}{E(s)}$$

→ Applying KCL at node v_1 ,

$$I_1 + I_2 + I_3 = 0$$

$$\left[\frac{v_1 - E(t)}{R_1} + C \frac{dv_1}{dt} + \frac{v_1}{R_2} \right] = 0$$

$$\frac{v_1 - E(t)}{R_1} + C \frac{dv_1}{dt} + \frac{v_1 - V_2}{R_2} = 0$$

$$\frac{v_1 - E(t)}{R_1} + C_1 \frac{dv_1}{dt} + \frac{v_1 - V_2}{R_2} = 0$$

$$\frac{v_1 - E(t)}{R_1} + C \frac{dv_1}{dt} + \frac{v_1 - V_2}{R_2} = \frac{E(t)}{R_1}$$

On taking L.T. of above equation with zero initial condition, we

get,

$$\frac{V_1(s)}{R_1} + C s V_1(s) + \frac{V_1(s)}{R_2} - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$\left[\frac{1}{R_1} + C s + \frac{1}{R_2} \right] V_1(s) - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1} \rightarrow ①$$

→ Applying KCL at node v_2

$$I_4 + I_5 = 0$$

$$\frac{V_2 - V_1}{R_2} + C_2 \frac{dV_2}{dt} = 0 \quad (1)$$

$$\frac{V_2}{R_2} - \frac{V_1}{R_2} + C_2 \frac{dV_2}{dt} = 0 \quad \text{emf type load model}$$

on taking L.T. of above equation with zero initial condition, we get

$$\frac{V_2(s)}{R_2} - \frac{V_1(s)}{R_2} + sC_2 V_2(s) = 0$$

$$\left[\frac{1}{R_2} + C_2(s) \right] V_2(s) - \frac{V_1(s)}{R_2} = 0 \quad (2)$$

$$V_1(s) = R_2 \left[\frac{1 + R_2 C_2(s)}{R_2} \right] V_2(s)$$

$$V_1(s) = [1 + R_2(s)] V_2(s) \quad (3)$$

sub eqn-(3) in eqn-(1)

$$\left[\frac{1}{R_1} + C_1(s) + \frac{1}{R_2} \right] \left[1 + R_2(s) \right] V_2(s) - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$\left[\frac{R_2 + C_1 R_1 R_2 + R_1}{R_1 R_2} \right] \left[V_2(s) + R_2 V_2(s) \right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$\left[\frac{R_2 + C_1 R_1 R_2 + R_1}{R_1 R_2} \right] \left[1 + R_2(s) \right] - \frac{1}{R_2} \frac{V_2(s)}{1} = \frac{E(s)}{R_1}$$

$$\left[R_1 + R_2 + R_1 R_2 C_1 \right] \left[1 + R_2(s) \right] - R_1 = \frac{E(s)}{R_1 R_2}$$

$$\therefore \frac{V_2(s)}{E(s)} = \frac{\frac{1}{R_2}}{\left[R_1 + R_2 + R_1 R_2 C_1 \right] \left[1 + R_2(s) \right] - R_1}$$

\Rightarrow Analogous Systems :-

They are 2 types :-

1) force voltage analogy.

Force voltage analog system

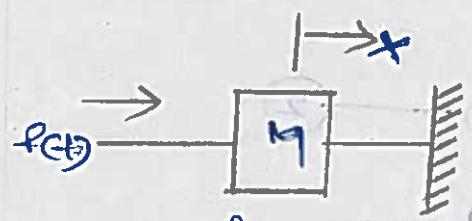
2) force current analogy.

1) Force Voltage Analogy :-

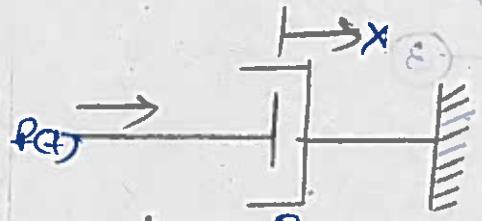
Mechanical Systems.

Input = force

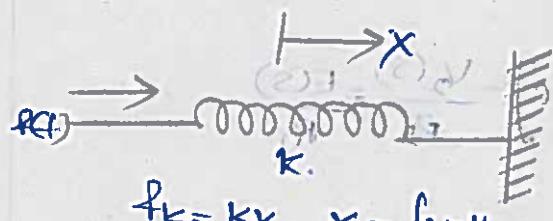
Output = Velocity.



$$f_m = M \frac{dx}{dt^2}$$



$$f_b = B \frac{dx}{dt} = BV$$

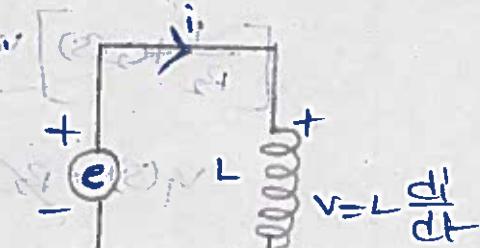


$$f_k = kx, \quad x = \int v dt$$

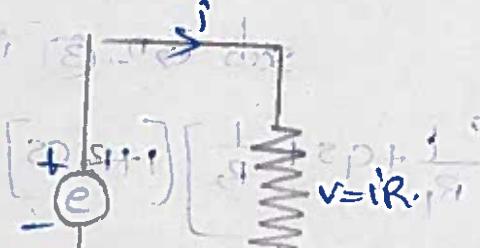
Electrical Systems.

Input = Voltage Source.

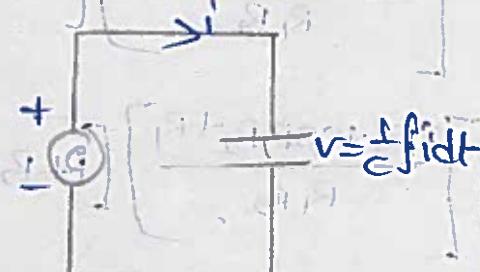
Output = current through the element.



$$V = L \frac{di}{dt}$$

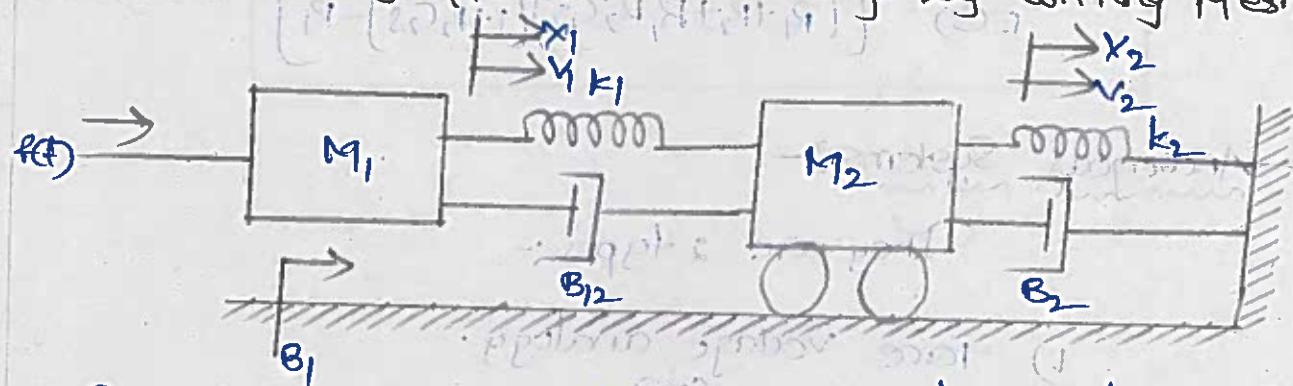


$$V = iR$$

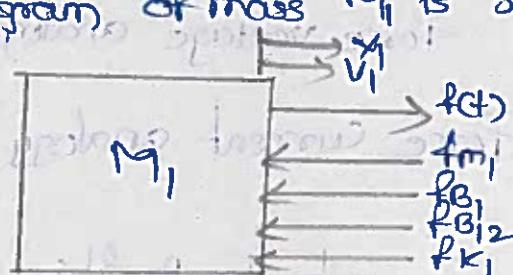


$$V = \frac{1}{C} \int i dt$$

- ① write differential equations governing the mechanical system shown in the figure. Draw the force voltage electrical analogy, ckt and velocity. by writing mesh eqn?



free body diagram of mass M_1 is shown below figure.



The force balanced equations are

(12)

$$f_{m_1} = M_1 \frac{d^2x_1}{dt^2}$$

$$f_{B_{12}} = B_1 \frac{d}{dt} [x_1 - x_2]$$

$$f_{B_1} = B_1 \frac{dx_1}{dt}$$

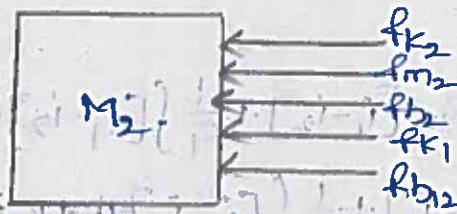
$$f_k_1 = k_1 [x_1 - x_2]$$

from Newton's second law,

$$f(t) = f_{m_1} + f_{B_{12}} + f_{B_1} + f_k_1$$

$$f(t) = M_1 \frac{d^2x_1}{dt^2} + B_{12} \frac{d}{dt} [x_1 - x_2] + B_1 \frac{dx_1}{dt} + k_1 [x_1 - x_2] \quad (1)$$

free body diagram of mass m_2 ,



The force balanced equations are,

$$f_{m_2} = M_2 \frac{d^2x_2}{dt^2}$$

$$f_{k_2} = k_2 x_2$$

$$f_{b_2} = B_2 \frac{dx_2}{dt}$$

$$f_{b12} = B_{12} \frac{d}{dt} [x_2 - x_1]$$

$$f_{b_{12}} = B_1 \frac{d}{dt} [x_2 - x_1]$$

from Newton's second law,

$$0 = f_{m_2} + f_{b_2} + f_{b_{12}} + f_k_2 + f_{k_2}$$

$$0 = M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12} \frac{d}{dt} [x_2 - x_1] + k_1 [x_2 - x_1] + k_2 x_2$$

$$0 = M_2 \frac{dv_2}{dt} + B_2 v_2 + B_{12} [v_2 - v_1] + k_1 \{ [v_2 - v_1] dt + k_2 \} v_2 dt \quad (2)$$

The electrical analogous elements for the mechanical system, are given by;

$$f(t) \rightarrow i(t)$$

$$v_1 \rightarrow i_1$$

$$v_2 \rightarrow i_2$$

$$M_1 \rightarrow L_1$$

$$M_2 \rightarrow L_2$$

$$B_{12} \rightarrow R_{12}$$

$$B_1 \rightarrow R_1$$

$$B_2 \rightarrow R_2$$

$$k_1 \rightarrow \frac{1}{C_1}$$

$$k_2 \rightarrow \frac{1}{C_2}$$

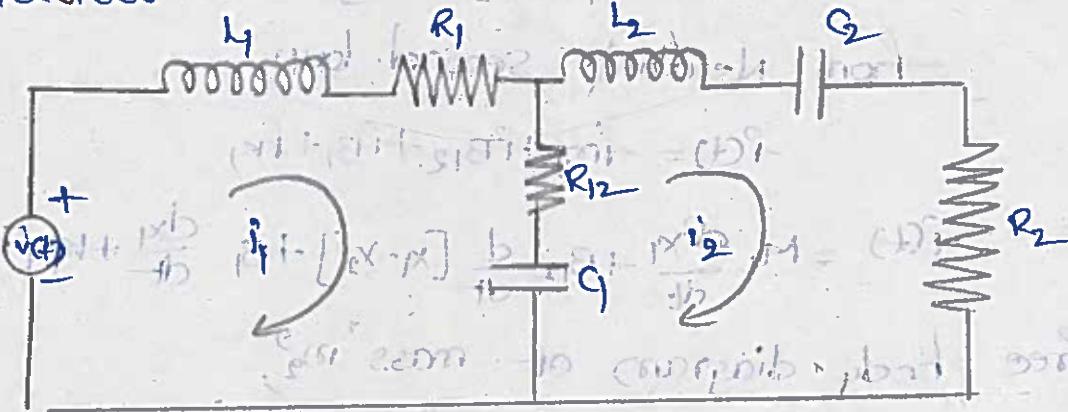
from Eq ① is given by;

$$v(t) = L_1 \frac{di}{dt} + R_{12} [i_1 - i_2] + \frac{1}{C_2} \int [i_1 - i_2] dt + R_1 i_1 \rightarrow ③$$

from - ②

$$0 = L_2 \frac{di_2}{dt} + R_{12} [i_2 - i_1] + R_2 i_2 + \frac{1}{C_1} \int [i_2 - i_1] dt + \frac{1}{C_2} \int i_2 dt \rightarrow ④$$

Electrical system :-



from ckt,

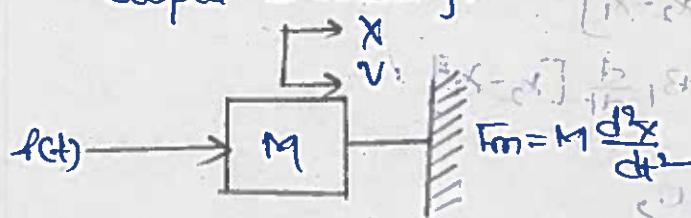
$$v(t) = L_1 \frac{di_1}{dt} + R_1 i_1 + R_{12} [i_1 - i_2] + \frac{1}{C_1} \int [i_1 - i_2] dt \rightarrow ⑤$$

$$0 = L_2 \frac{di_2}{dt} + R_{12} [i_2 - i_1] + \frac{1}{C_1} \int [i_2 - i_1] dt + R_2 i_2 + \frac{1}{C_2} \int i_2 dt \rightarrow ⑥$$

2) force current Analogy :-

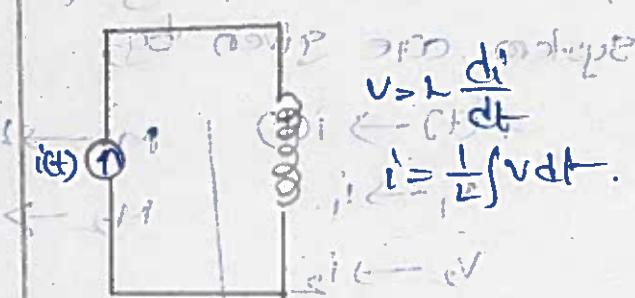
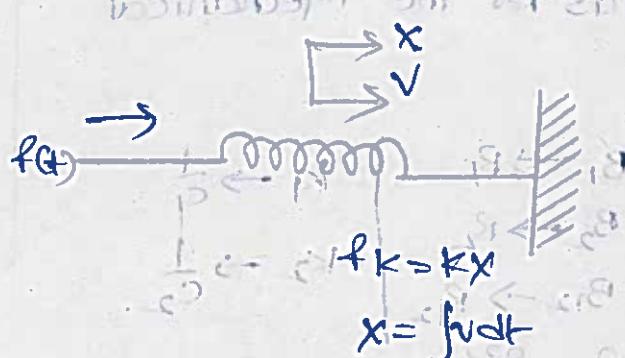
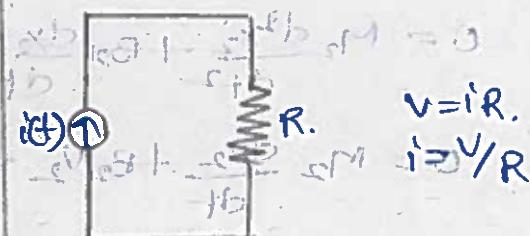
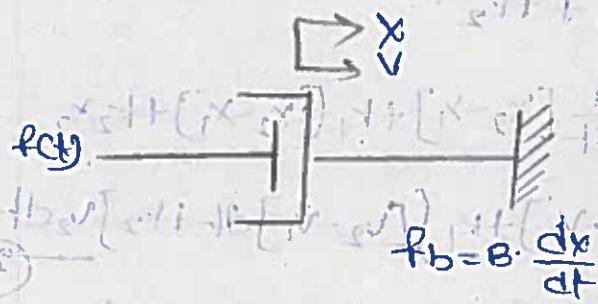
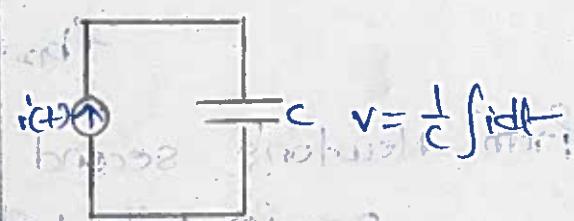
Mechanical Systems.

Input = force
output = velocity.

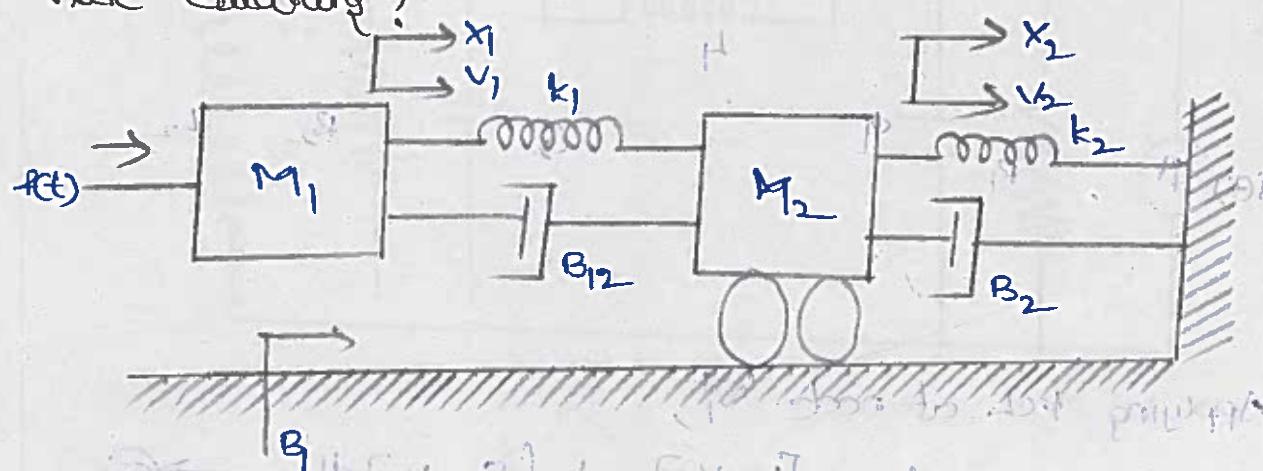


Electrical Systems.

Input = current source
output = voltage source



① write the differential equation governing the mechanical system shown in fig. Draw the force current electrical analogous ckt and verify by writing node equations? (B)



Ans:- The differential eqn for mass M_1

$$f(t) = M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}[x_1 - x_2] + k_1 [x_1 - x_2]$$

$$f(t) = M_1 \frac{dv_1}{dt} + B_1 v_1 + B_{12} [v_1 - v_2] + k_1 [v_1 - v_2] dt \rightarrow 1$$

The D.E for mass M_2 ,

$$0 = M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12} \frac{d}{dt}[x_2 - x_1] + k_2 x_2 + k_1 [x_2 - x_1]$$

$$0 = M_2 \frac{dv_2}{dt} + B_2 v_2 + B_{12} [v_2 - v_1] + k_2 \int v_2 dt + k_1 \int [v_2 - v_1] \rightarrow 2$$

The electrical analogous elements for the mechanical system are given by;

$$f(t) \rightarrow i(t) \quad M_1 \rightarrow C_1$$

$$v \rightarrow v_{ab} \quad M_2 \rightarrow C_2$$

$$v_2 \rightarrow v_2$$

$$B_1 \rightarrow \frac{1}{R_1}$$

$$B_2 \rightarrow \frac{1}{R_2}$$

$$B_{12} \rightarrow \frac{1}{R_{12}}$$

$$k_1 \rightarrow \frac{1}{L_1}$$

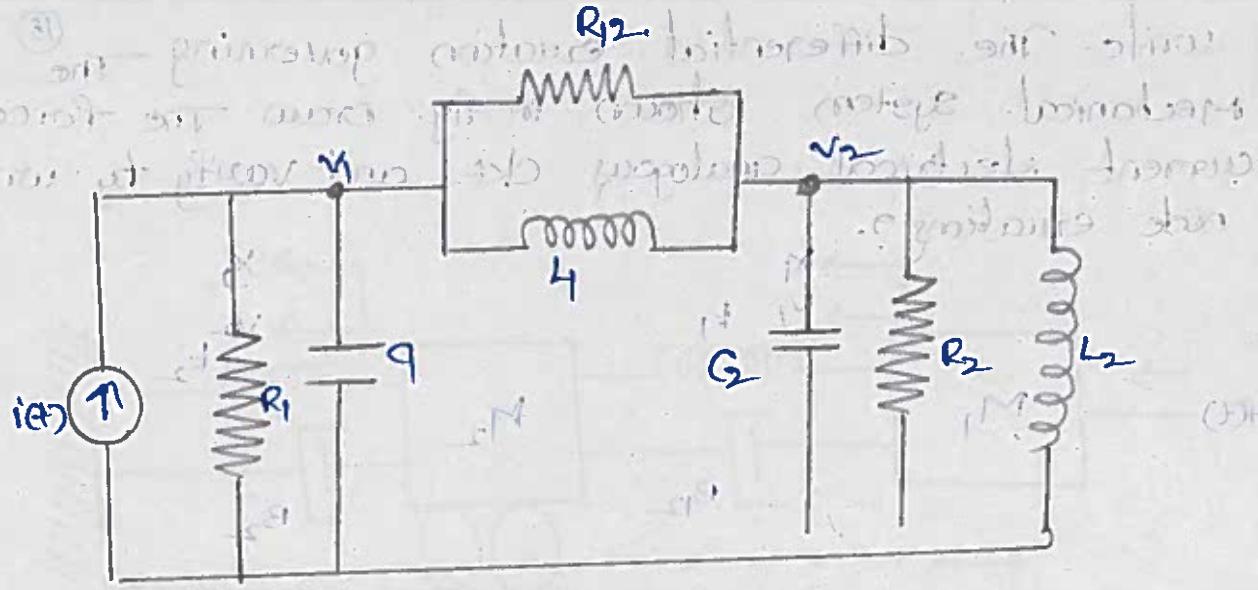
$$k_2 \rightarrow \frac{1}{L_2}$$

from $\rightarrow 1$.

$$i(t) = C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{(v_2 - v_1)}{R_{12}} + \frac{1}{L_1} \int [v_1 - v_2] dt \rightarrow 3$$

from $\rightarrow 2$

$$0 = C_2 \frac{dv_2}{dt} + \frac{v_2}{R_2} + \frac{[v_2 - v_1]}{R_{12}} + \frac{1}{L_2} \int v_2 dt + \frac{1}{L_1} \int [v_2 - v_1] dt \rightarrow 4$$



Applying KCL at node 'N',

$$i(t) = \frac{V_1}{R_1} + C_1 \frac{dv_1}{dt} + \frac{[V_1 - V_2]}{R_{12}} + \frac{1}{L_1} \int [V_1 - V_2] dt \rightarrow \textcircled{5}$$

Applying KCL at node 'V2'

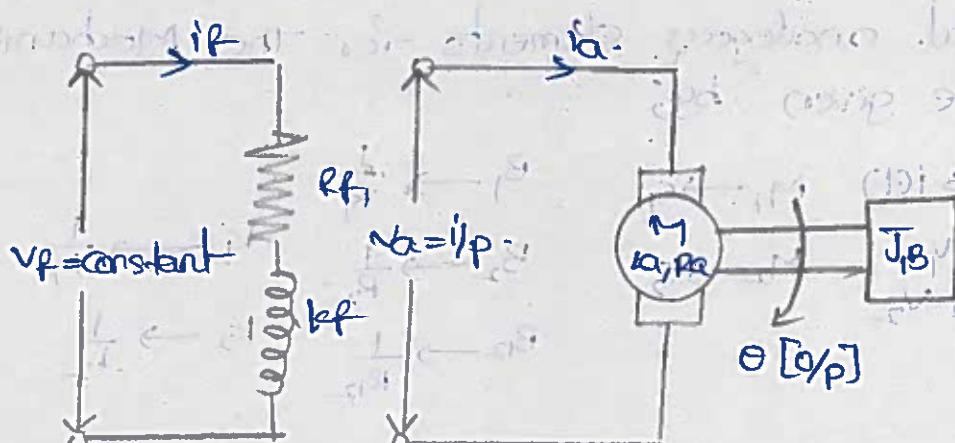
$$0 = G_2 \frac{dv_2}{dt} + \frac{V_2}{R_2} + \frac{1}{L_2} \int v_2 dt + \frac{[V_2 - V_1]}{R_{12}} + \frac{1}{L_1} \int [V_2 - V_1] dt \rightarrow \textcircled{6}$$

$$\therefore \text{Eqn } \textcircled{5} = \text{Eqn } \textcircled{6}$$

$$\text{Eqn } \textcircled{4} = \text{Eqn } \textcircled{6}$$

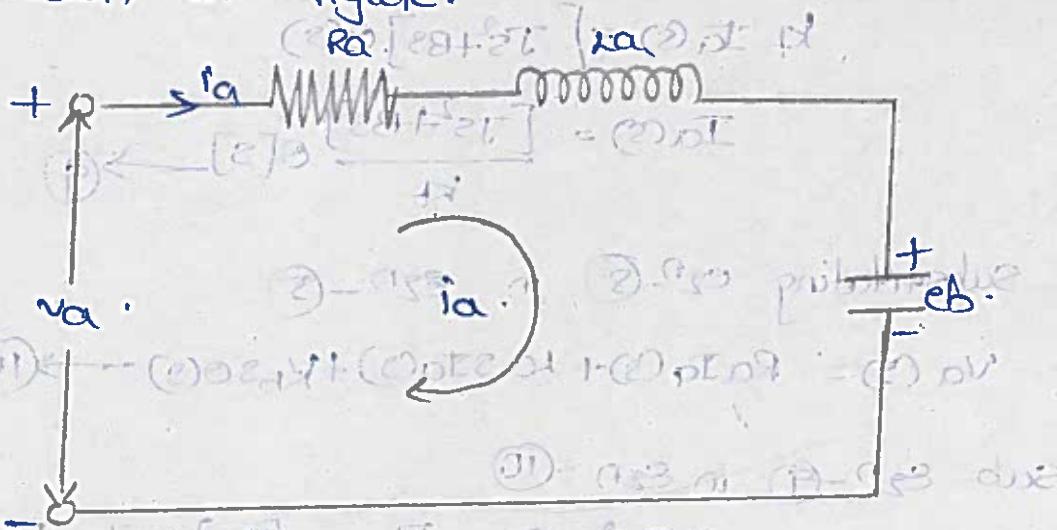
Hence, Verified node eqn

\Rightarrow transfer function of armature controlled DC motor
(or) DC servomotor:-



- * The speed of DC motor is directly proportional to armature voltage and inversely proportional to flux.
- * In armature controlled DC motor, the desired speed is obtained by varying the armature voltage.
- * This speed control system is an electro mechanical system. The electrical system consisting of armature ckt and field ckt, but for analysis purpose only the armature ckt is considered, because the field excited by a constant voltage source.

* The equivalent ckt of armature of dc motor is shown in figure.



applying KVL to the above ckt,

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + e_b \rightarrow ①$$

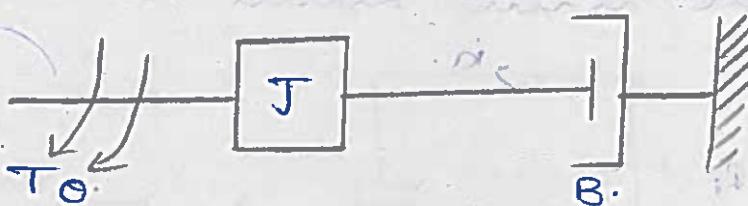
The Torque of the DC Motor is proportional to product of flux and current.

$$T \propto \phi i_a$$

$$T \propto i_a \quad [\because \text{flux is constant}]$$

$$T = k_i i_a \rightarrow ②$$

The equivalent mechanical system is given by;



from Newton's second law,

$$T = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} \rightarrow ③$$

The back emf of DC Motor is directly proportional to angular velocity of the body.

$$e_b \propto \frac{d\theta}{dt}$$

$$e_b = k_b \cdot \frac{d\theta}{dt} \rightarrow ④$$

on taking L.T of above eqn ④, with zero initial conditions, we get

$$V_a(s) = R_a i_a(s) + k_b s i_a(s) + E_b(s) \rightarrow ⑤$$

$$T(s) = k_i i_a(s) \rightarrow ⑥$$

$$T(s) = J s^2 \theta(s) + B s \theta(s) \rightarrow ⑦$$

$$E_b(s) = k_b s \theta(s) \rightarrow ⑧$$

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Equating the eqns (6) & (7)

$$k_f I_a(s) = [Js^2 + Bs] \theta(s)$$

$$I_a(s) = \frac{[Js^2 + Bs]}{k_f} \theta(s) \rightarrow (9)$$

Substituting eqn - (8) in eqn - (5)

$$V_a(s) = R_a I_a(s) + L_a s I_a(s) + K_b s \theta(s) \rightarrow (10)$$

Sub eqn - (9) in eqn - (10)

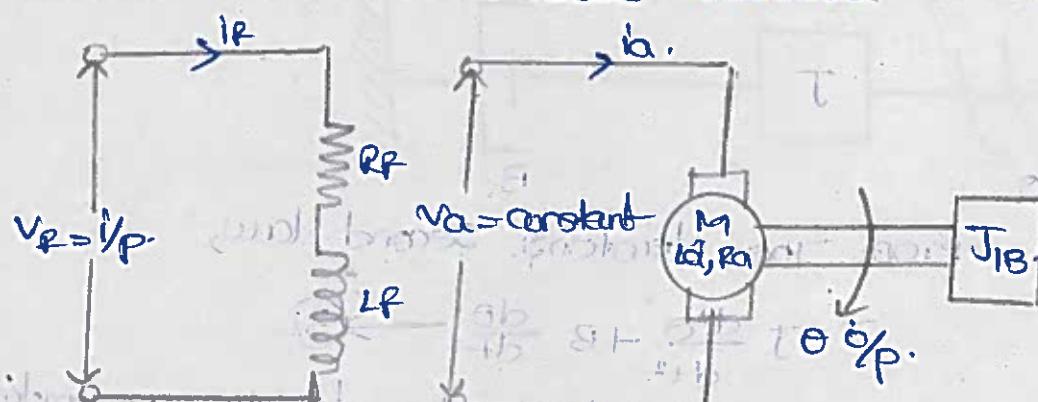
$$V_a(s) = R_a \left[\frac{[Js^2 + Bs]}{k_f} \theta(s) \right] + L_a s \left[\frac{[Js^2 + Bs]}{k_f} \theta(s) \right] + K_b s \theta(s)$$

$$V_a(s) = [R_a + L_a s] \left[\frac{[Js^2 + Bs]}{k_f} \theta(s) \right] + K_b s \theta(s)$$

$$V_a(s) = \left[\frac{[R_a + L_a s][Js^2 + Bs] + K_b k_f s}{k_f} \theta(s) \right]$$

$$\therefore \frac{\theta(s)}{V_a(s)} = \frac{k_f}{[R_a + L_a s][Js^2 + Bs] + K_b k_f s}$$

\Rightarrow Transfer function of field controlled DC motor:-

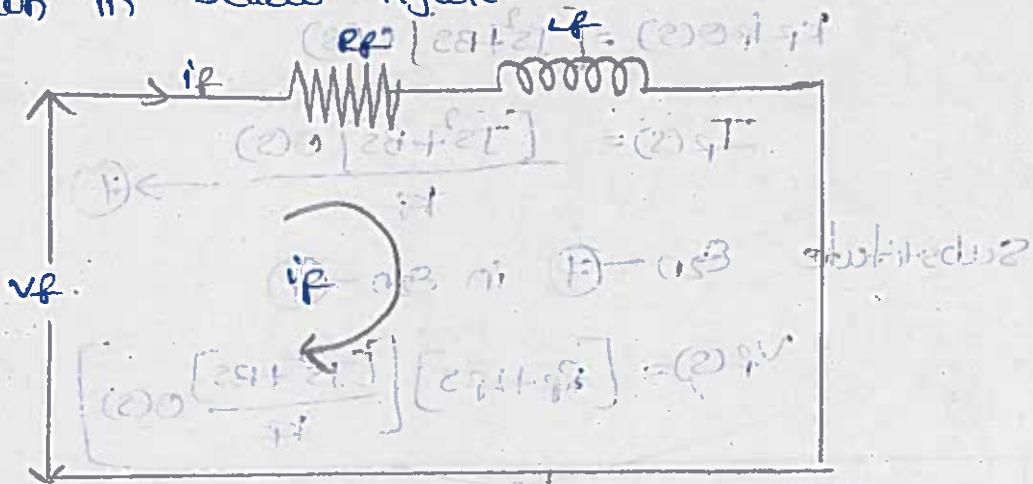


* The speed of DC motor is directly proportional to armature voltage and inversely proportional to flux.

* In field controlled DC motor, the desired speed is obtained by varying the field voltage.

* The control system or electromechanical system consists of electrical system consisting armature ckt, field ckt but for analysis purpose only field ckt is considered because the armature ckt is excited by a constant voltage source.

* The equivalent ckt of field control of D.C motor is shown in bellow figure. (15)



applying KVL to the above ckt.

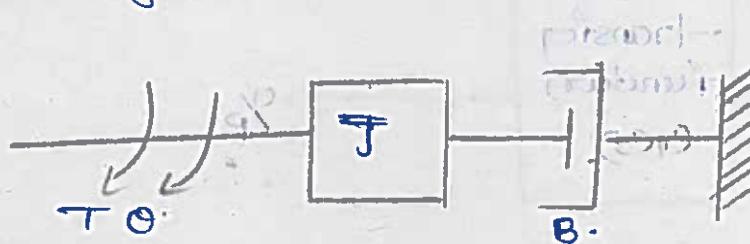
$$V_f = R_f i_f + L_f \frac{di_f}{dt} \rightarrow 1$$

The torque of the DC Motor is proportional to product of flux and current

$$T \propto \Phi i_f$$

$$T = k_f i_f \rightarrow 2$$

The equivalent ckt of Mechanical system as shown in the bellow figure.



from Newton's second law,

$$T = J \frac{d^2\theta}{dt^2} + B \cdot \frac{d\theta}{dt} \rightarrow 3$$

on taking L.T of above eqn 3 with zero initial condition

$$V_f(s) = R_f I_f(s) + L_f s I_f(s) \rightarrow 4$$

$$T(s) = k_f i_f(s) \rightarrow 5$$

$$T(s) = JS^2\theta(s) + BS\theta(s) \rightarrow 6$$

Equating eqns - 5 & 6.

$$k_f i_f(s) = Js^2 \Theta(s) + Bs\Theta(s)$$

$$k_f i_f(s) = [Js^2 + Bs] \Theta(s)$$

$$I_f(s) = \frac{[Js^2 + Bs] \Theta(s)}{k_f} \rightarrow \textcircled{f}$$

Substitute $\Theta_n - \textcircled{f}$ in $\Theta_n - \textcircled{d}$

$$V_f(s) = [R_f + L_f s] \left[\frac{[Js^2 + Bs]}{k_f} \Theta(s) \right]$$

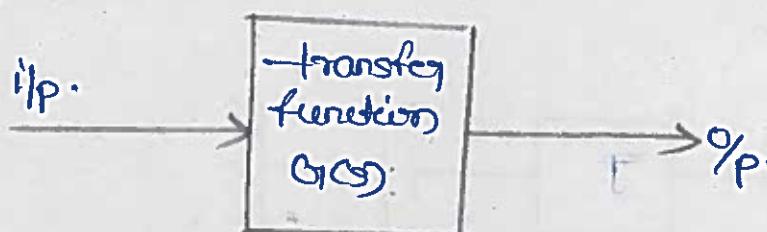
$$\therefore T_o f = \frac{\Theta(s)}{V_f(s)} = \frac{k_f}{[R_f + L_f s][Js^2 + Bs]}$$

\Rightarrow Block Diagram :-

- * The block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.

- * The elements of a block diagram are block, branch-point and summing point.

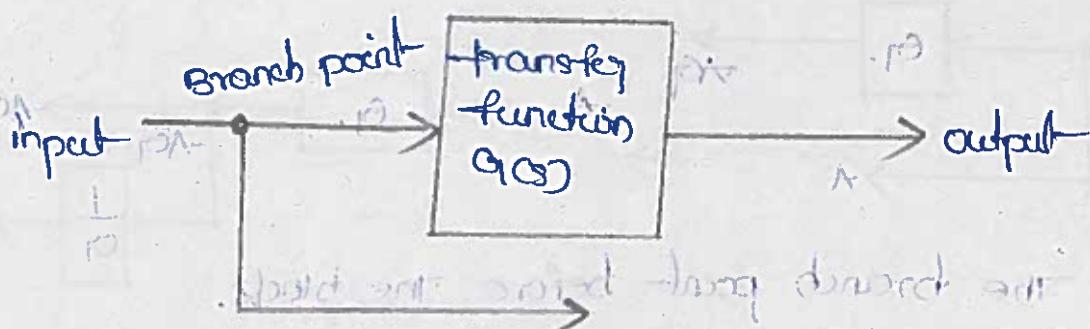
\Rightarrow Block :-



- * In block diagram all system variables are linked to each other functional blocks.

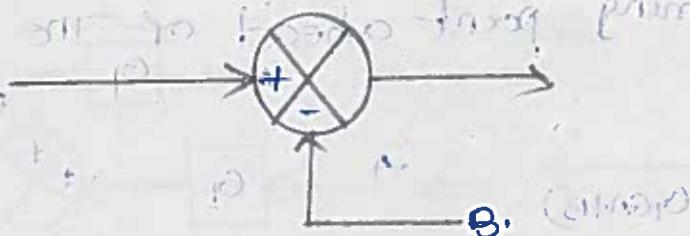
- * The arrowhead pointing towards the block indicates the input and the arrow head leading away from the block represents the output.

⇒ Branch point :- (16)



- * A branch point is a point from which the signal from a block goes concurrently to each other blocks.
(or) Summing point.

⇒ Summing points :-



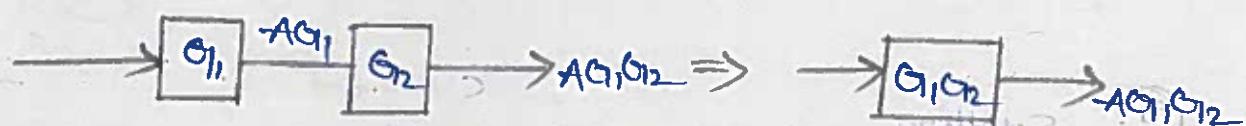
- * Summing points are used to add two (or) more signals in the system. A circle with cross is the symbol that indicates a summing operation.

- * The + sign at each arrow head indicates whether the signal is to be added (or) subtracted.

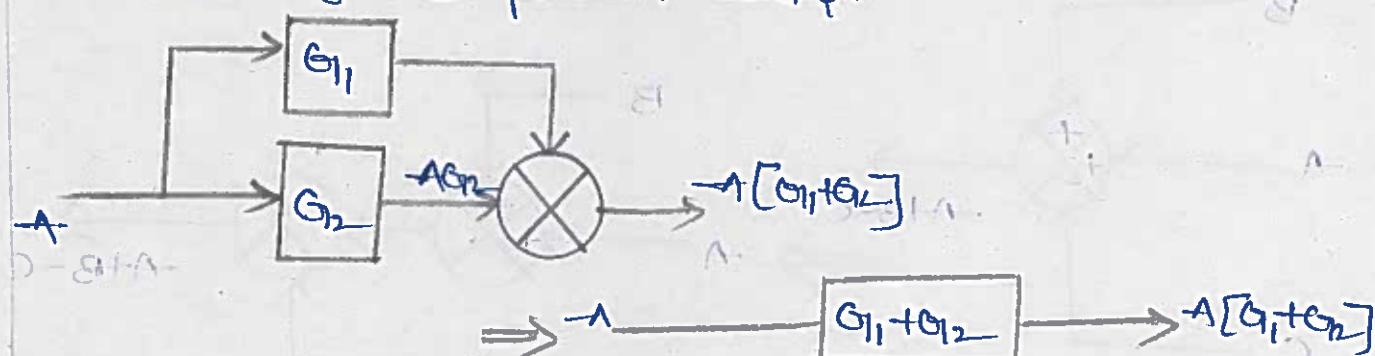
⇒ Block diagram reduction techniques :-

Rules :-

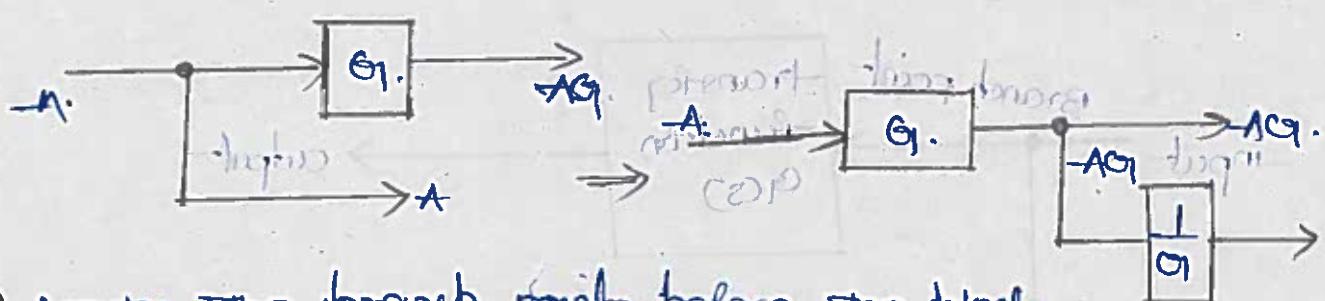
- ① Combining the blocks in cascade.



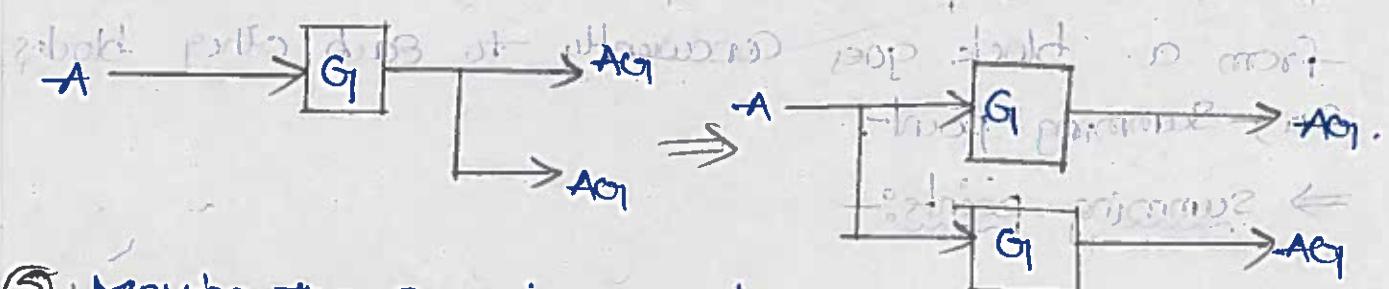
- ② Combining the parallel blocks.



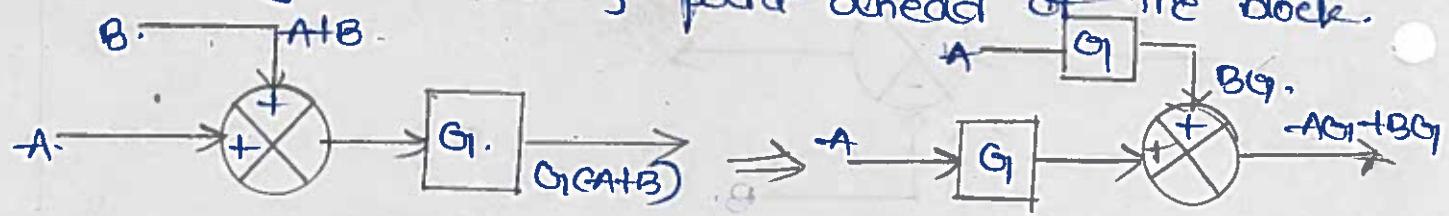
③ Moving the branch point ahead of the block.



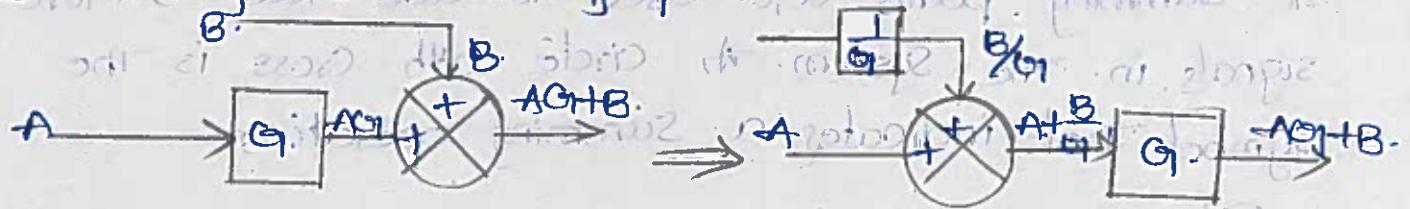
④ Moving the branch point before the block.



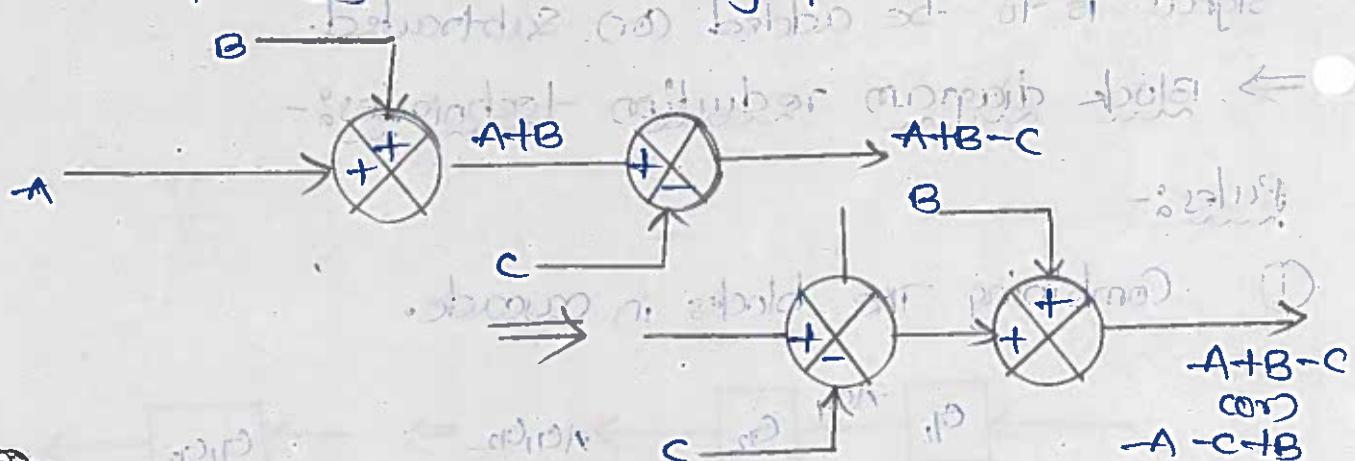
⑤ Moving the summing point ahead of the block.



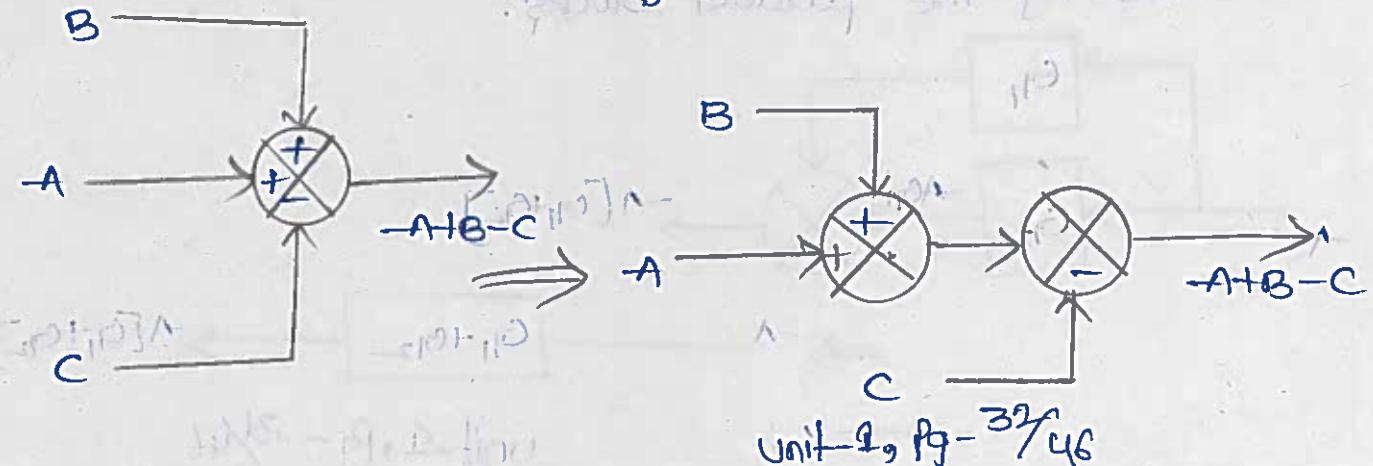
⑥ Moving the summing point before the block.



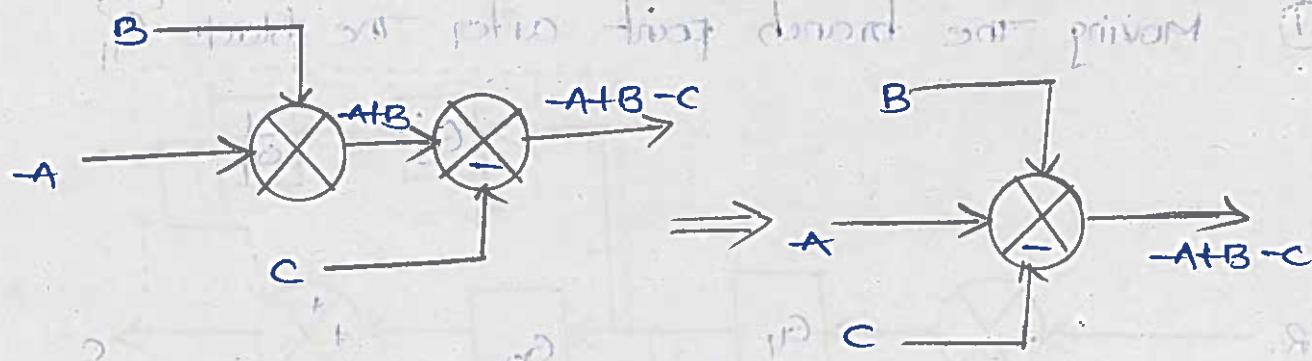
⑦ Interchanging the summing points.



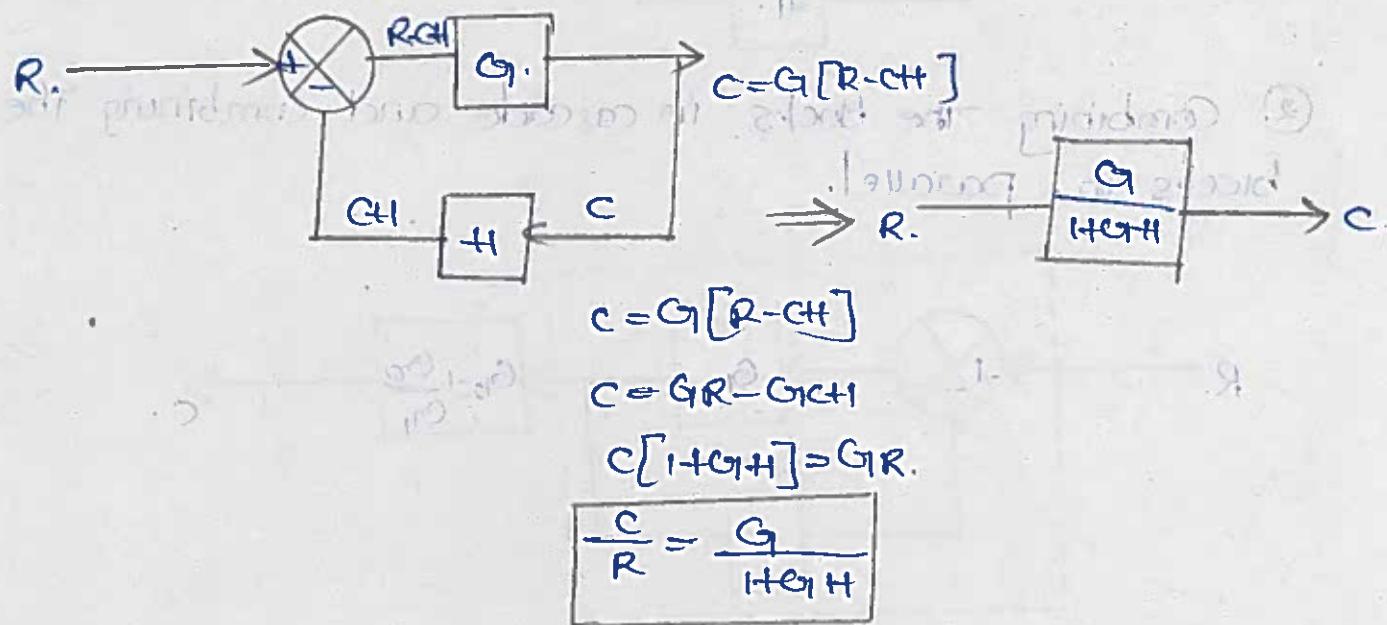
⑧ Splitting the summing points.



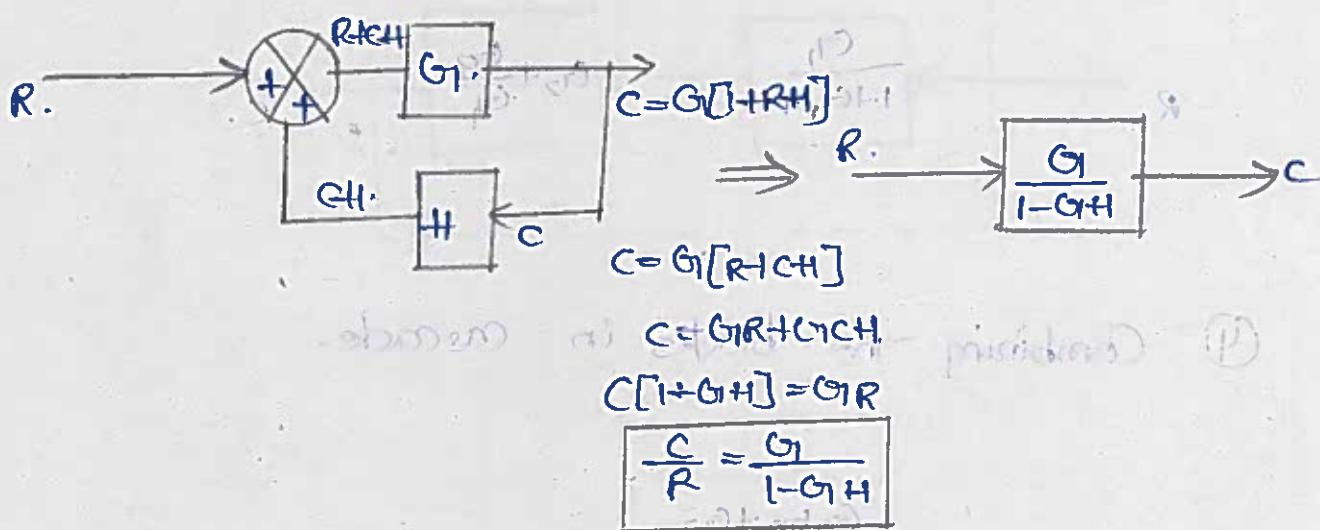
⑨ Combining the summing points.



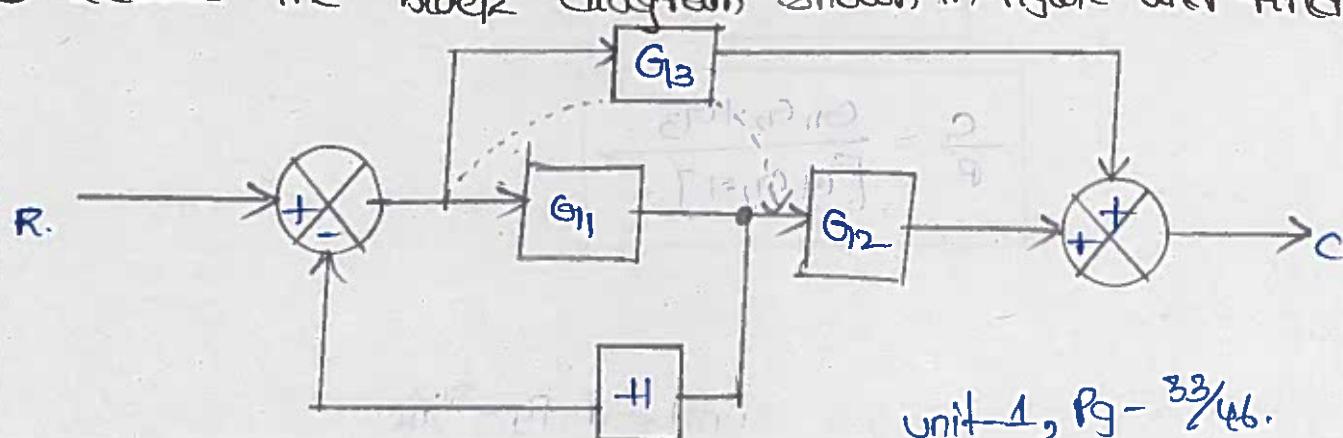
⑩ Elimination of feedback loop [-ve]



⑪ Elimination of +ve feedback loop

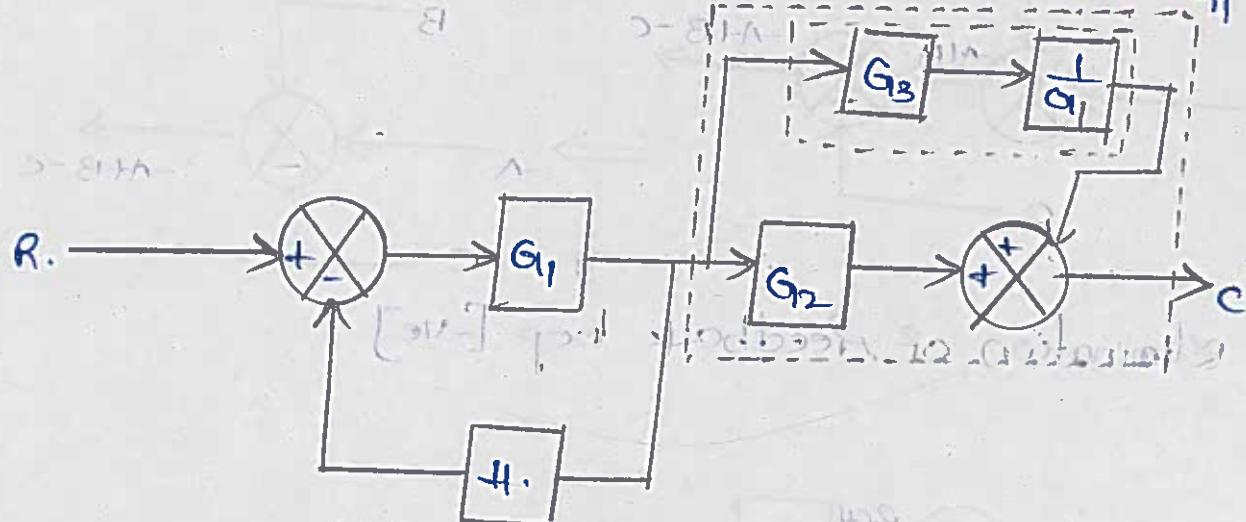


⑫ Reduce the block diagram shown in figure and find $\frac{C}{R}$?

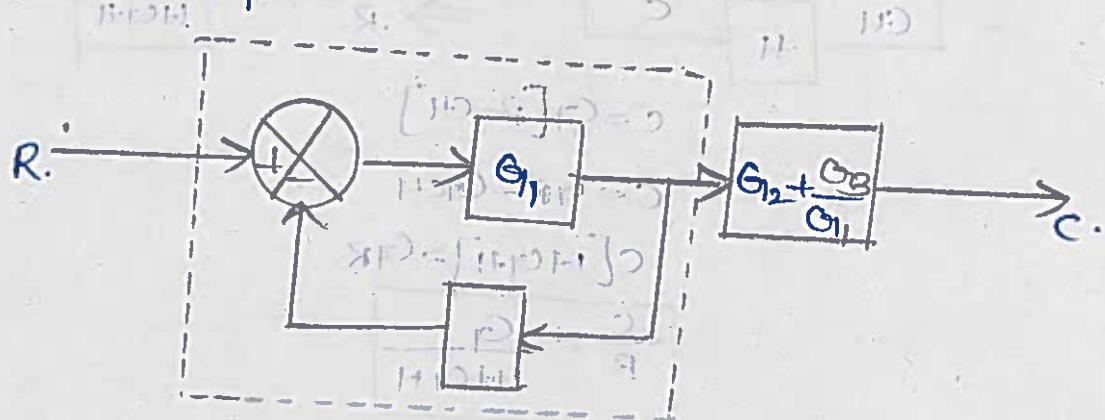


Step 1 Steps:-

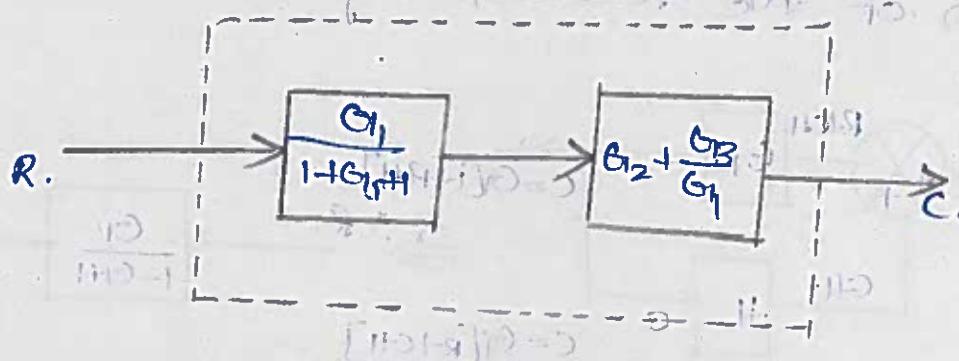
- ① Moving the branch point after the block G_1 .



- ② Combining the blocks in cascade and combining the blocks in parallel.



- ③ Elimination of '-ve' feedback.

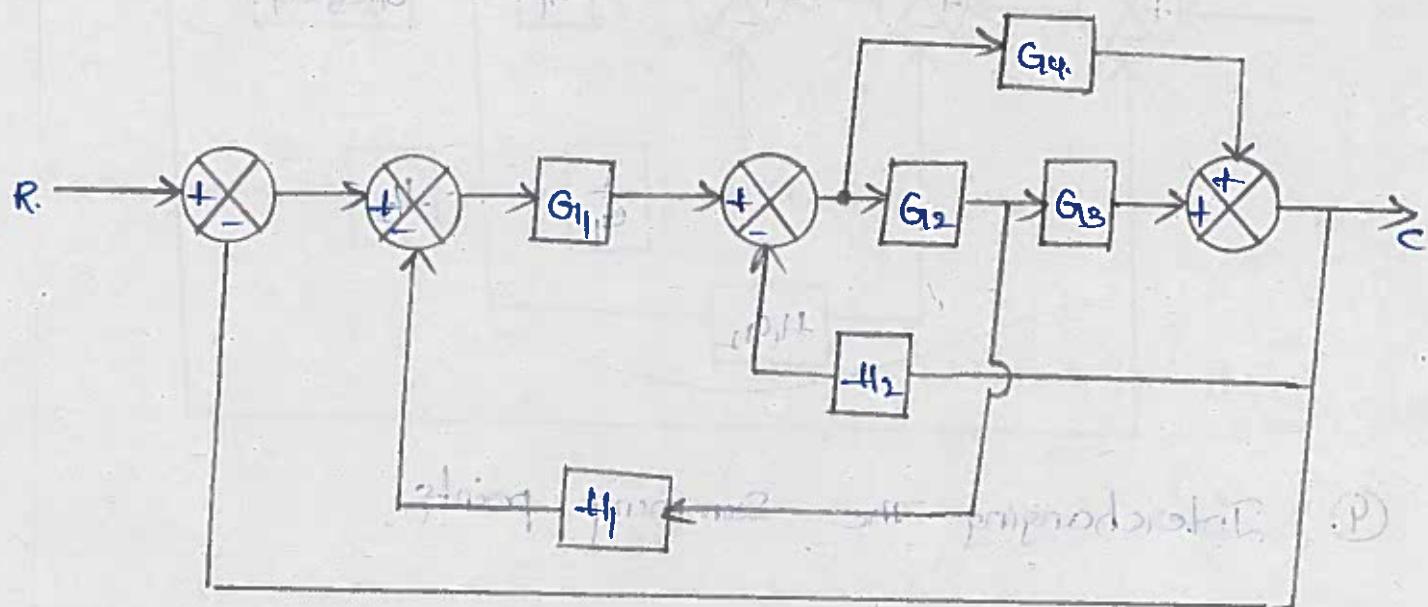


- ④ Combining the blocks in cascade.

Block diagram illustrating the final simplified block:

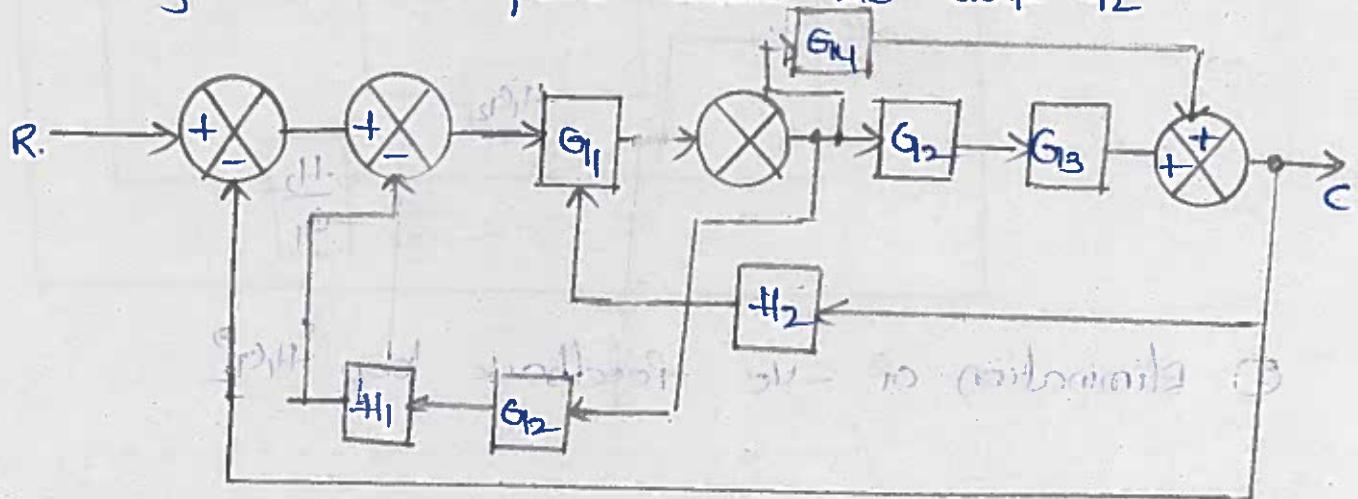
$$\frac{C}{R} = \frac{G_1 G_2 + G_1 G_3}{1 + G_1 + H}$$

Q) Using block diagram reduction techniques, find closed loop transfer function of the system whose block diagram is shown in figure? (18)

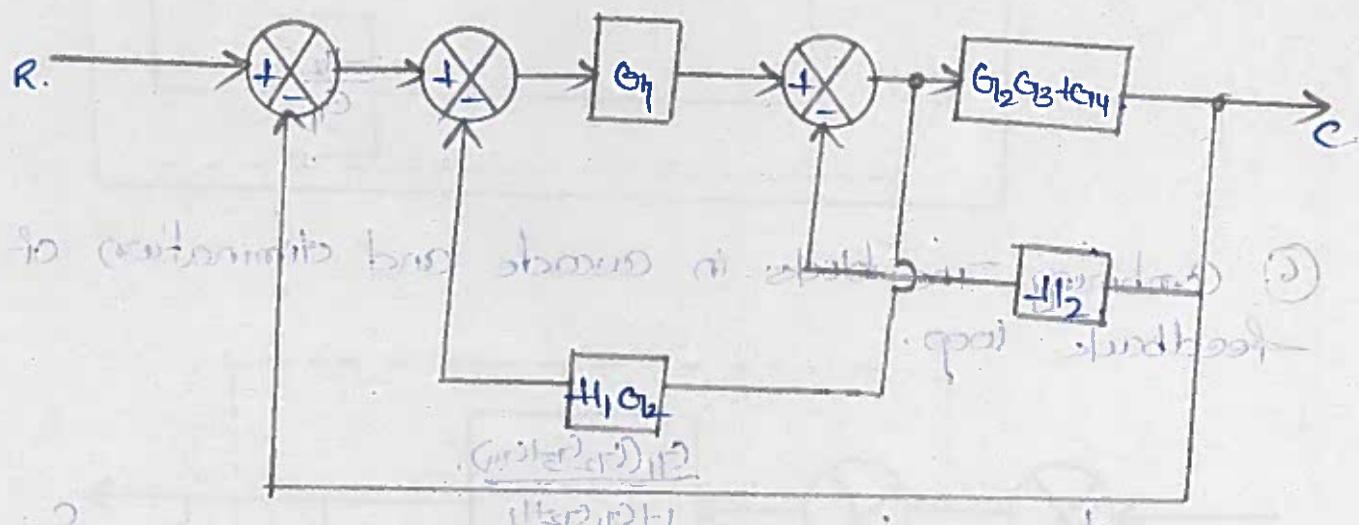


Solution steps:-

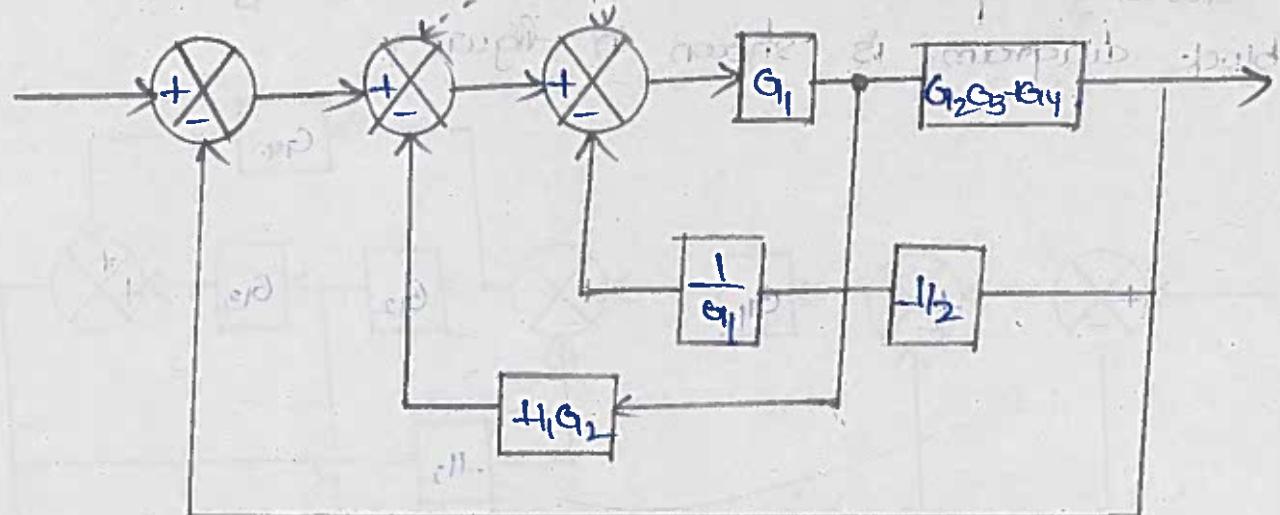
① Moving the branch point before the block G_2



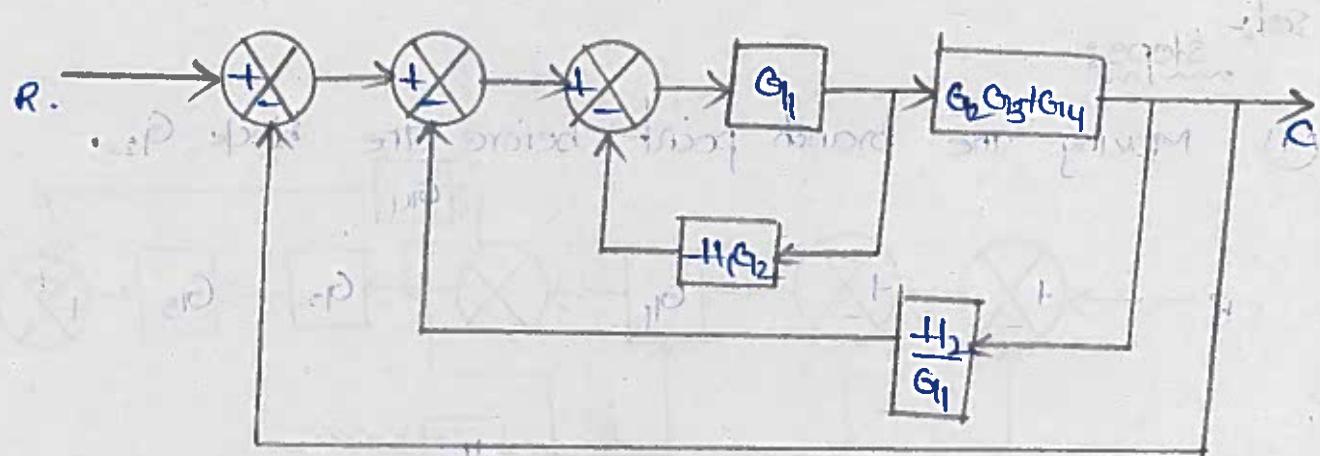
② Combining the blocks in cascade and parallel.



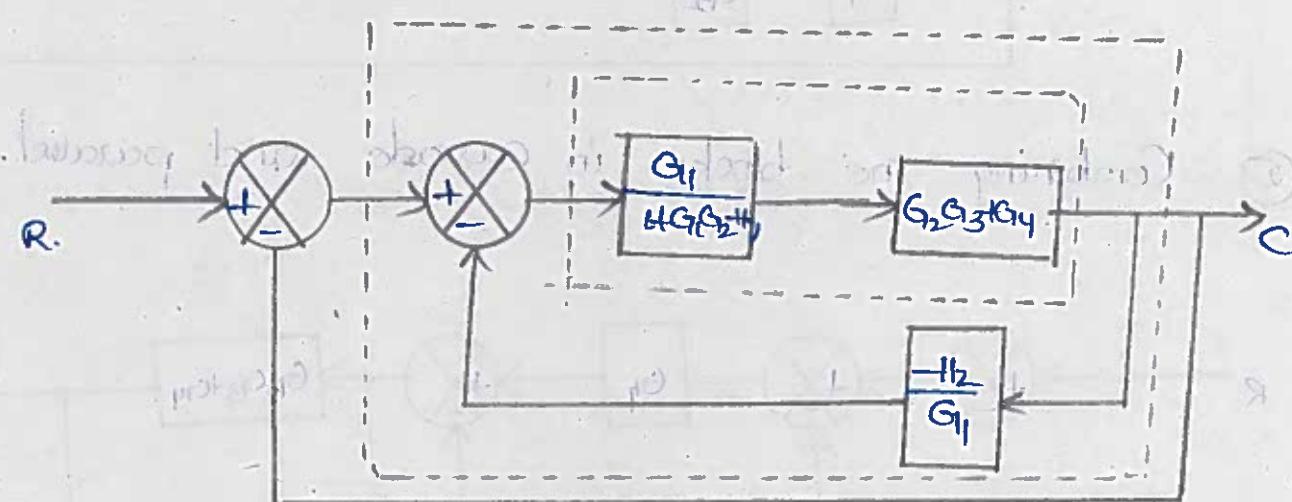
③ Moving the summing point before the block G_1 .



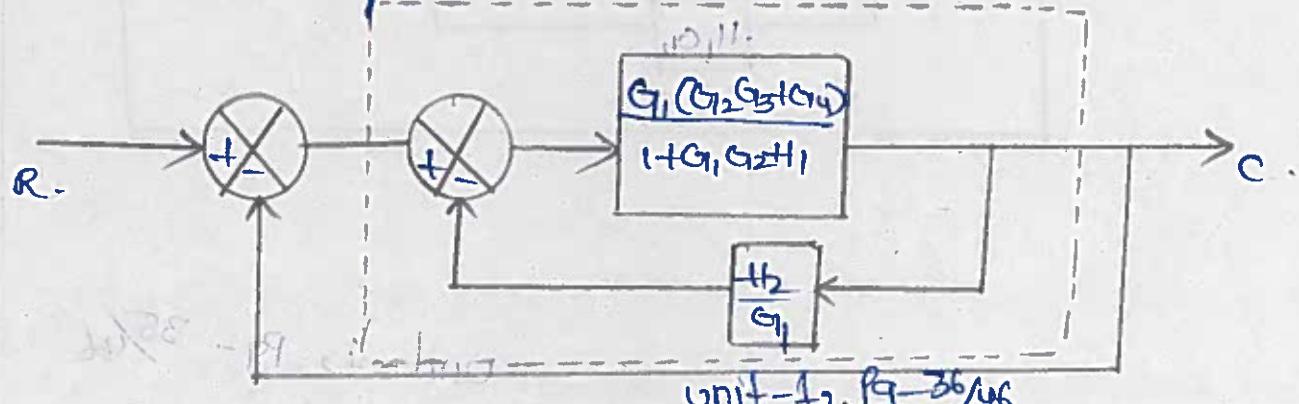
④ Interchanging the summing points.



⑤ Elimination of -ve feedback loop H_1G_2



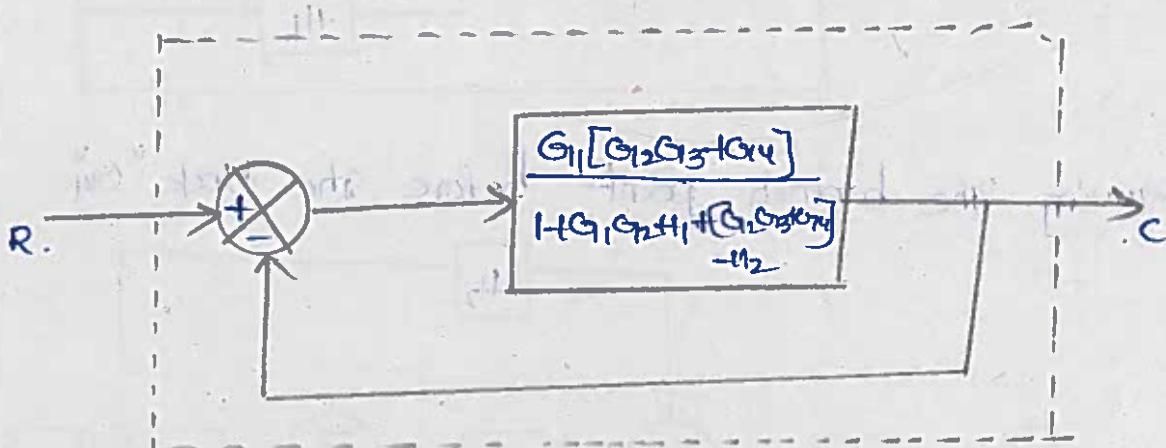
⑥ Combining the blocks in cascade and elimination of feedback loop.



$$\frac{G_1}{1+G_1H_1} = \frac{G_1[G_2G_3 + G_4]}{1 + G_1G_2H_1}$$

$$+ \frac{G_1[G_2G_3 + G_4]}{1 + G_1G_2H_1} \times \frac{H_2}{G_1}$$

$$\frac{G_1}{1+G_1H_1} = \frac{G_1[G_2G_3 + G_4]}{1 + G_1G_2H_1 + [G_2G_3 + G_4]H_2}$$



Elimination of feedback :-

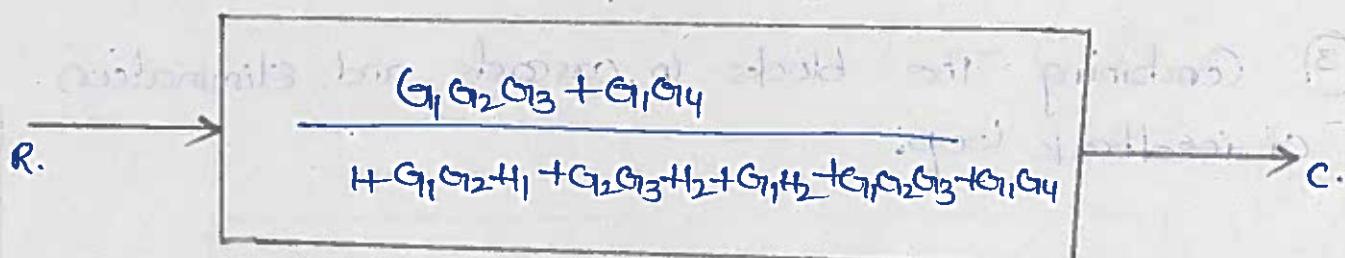
$$\frac{C}{R} = \frac{A}{1+A}$$

$$= \frac{G_1[G_2G_3 + G_4]}{1 + G_1G_2H_1 + [G_2G_3 + G_4]H_2}$$

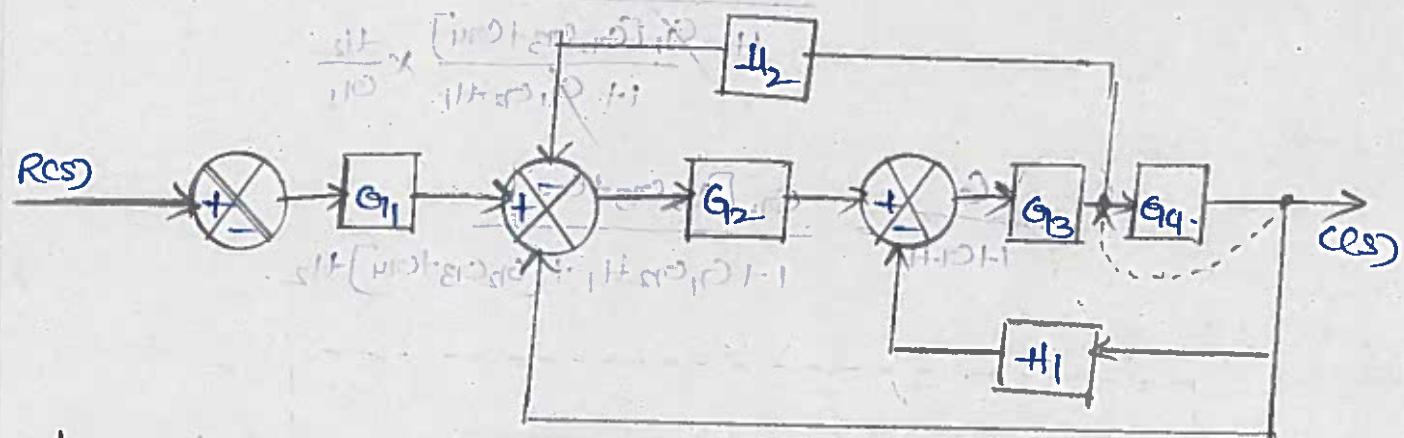
$$+ \frac{G_1[G_2G_3 + G_4]}{1 + G_1G_2H_1 + [G_2G_3 + G_4]H_2}$$

$$\frac{C}{R} = \frac{G_1[G_2G_3 + G_4]}{1 + [G_1G_2H_1 + G_2G_3H_2 + G_1H_2 + G_1G_2G_3 + G_1G_4]}$$

$$\frac{C}{R} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1H_2 + G_1G_2G_3 + G_1G_4}$$

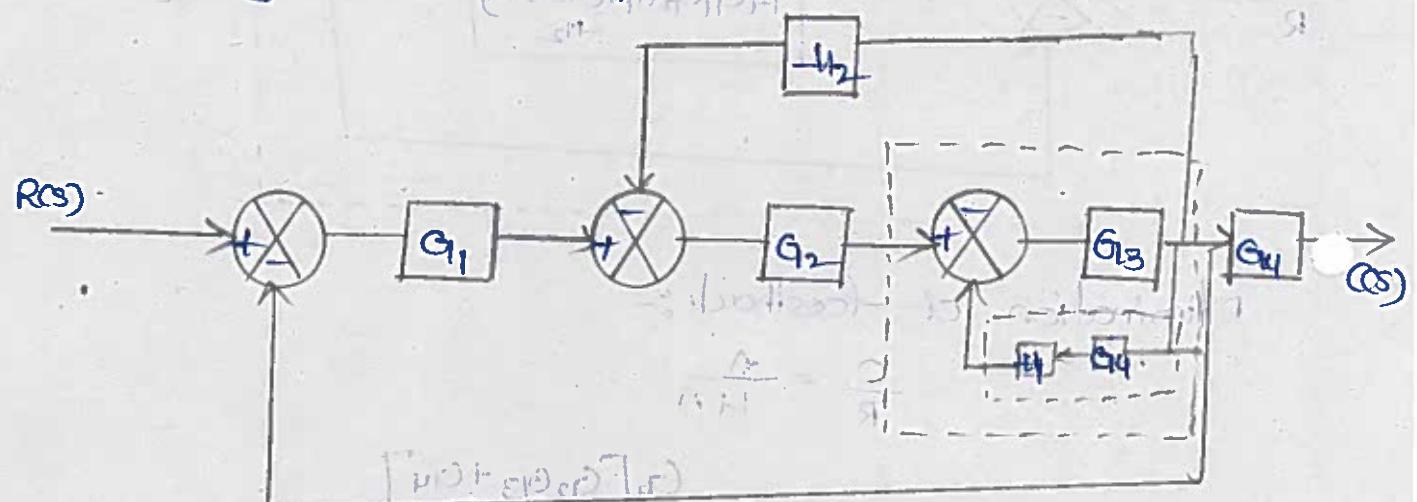


(c) determine the overall transfer function $\frac{C(s)}{R(s)}$ for the system shown in figure?

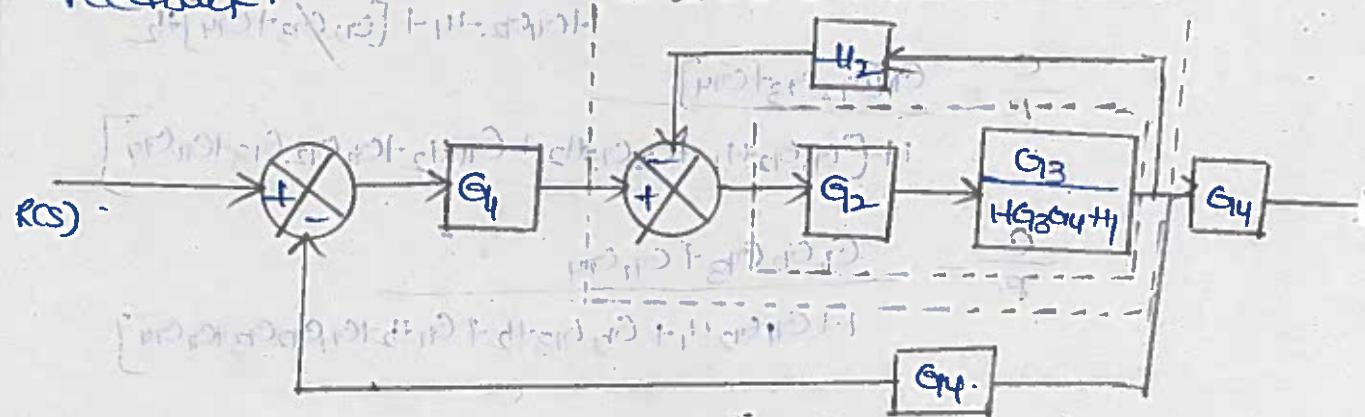


Step 1

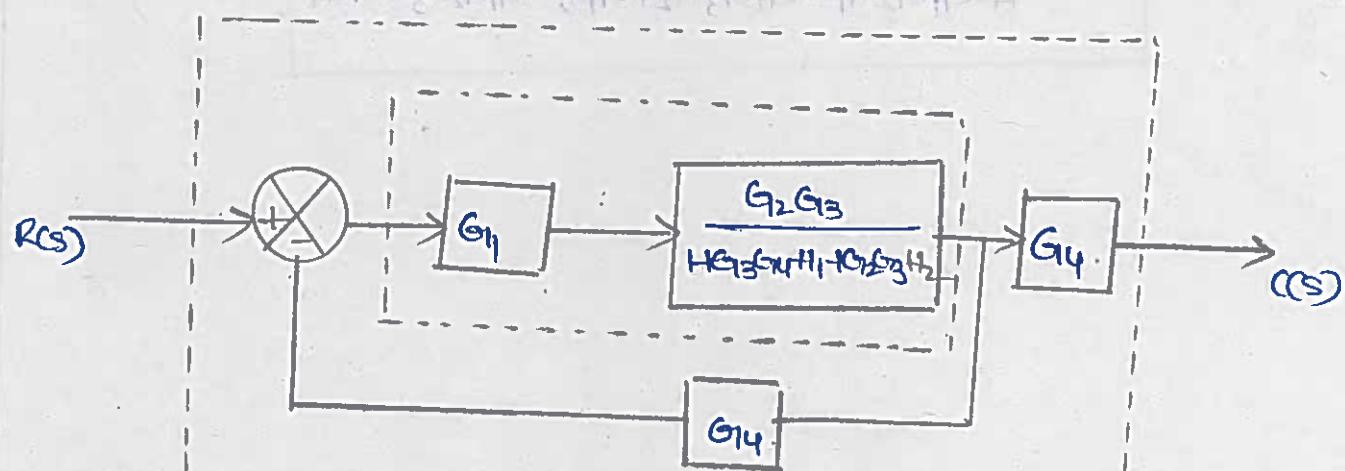
① Moving the branch point before the block "G4"



② Combining the blocks in cascade and elimination of feedback.



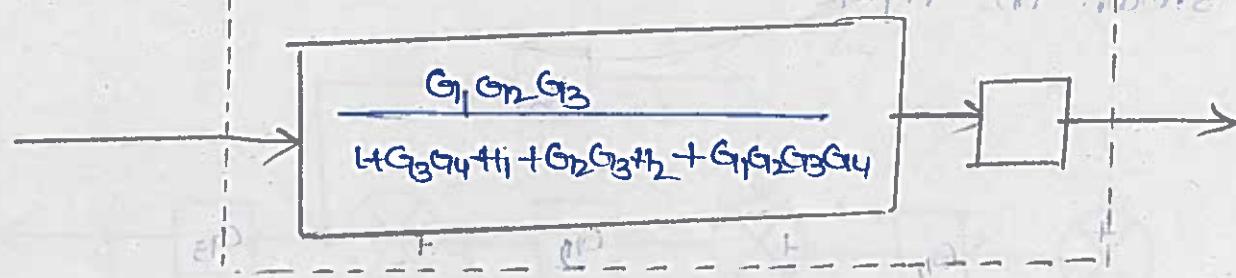
③ Combining the blocks in cascade and elimination of feedback loop.



④ Combining The blocks in cascade and elimination of feedback.

$$\frac{G}{H_1 H_2 H_3} = \frac{G_1 G_2 G_3}{1 + G_3 H_1 + G_2 H_3 H_1 + G_1 H_2 H_3}$$

$$\frac{G}{H_1 H_2 H_3 H_4} = \frac{G_1 G_2 G_3}{1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 H_4}$$



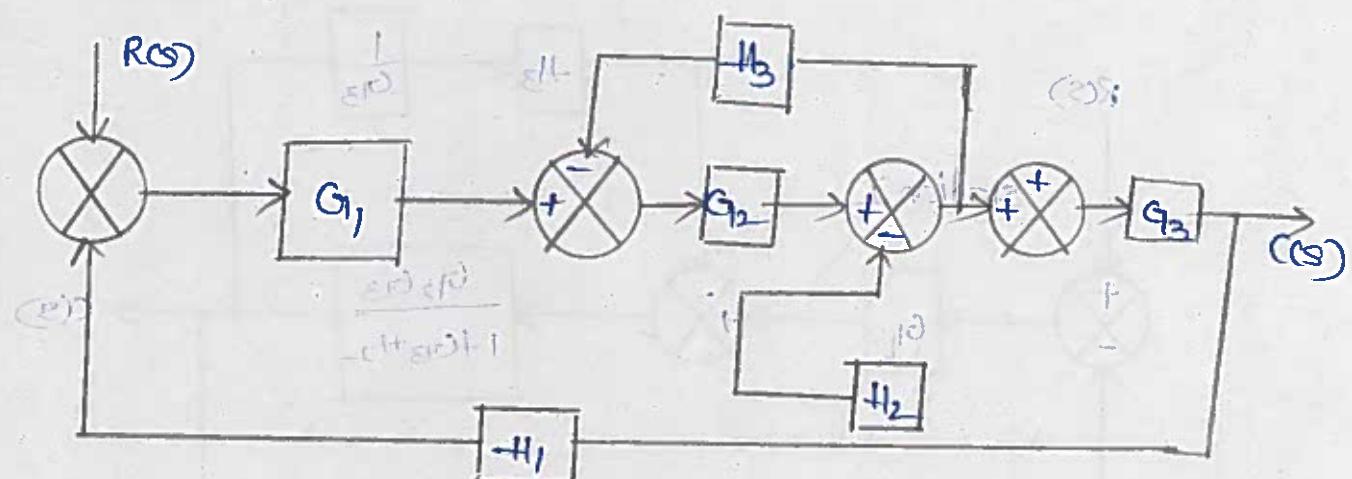
⑤ Combining The blocks in Cascade.

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 H_4} \rightarrow C(s)$$

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 H_4}$$

④ for The system represented by The block diagram shown in fig. Evaluate The closed loop transfer function when The input 'R' is, (i) At station -I (ii) At station -II

Station - I

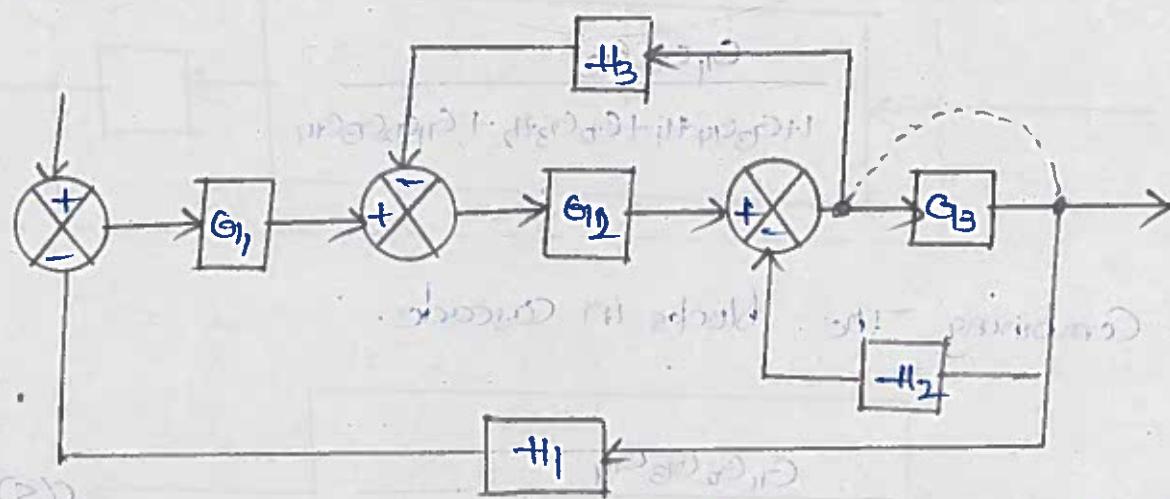


Consider the W_p at station - I :-

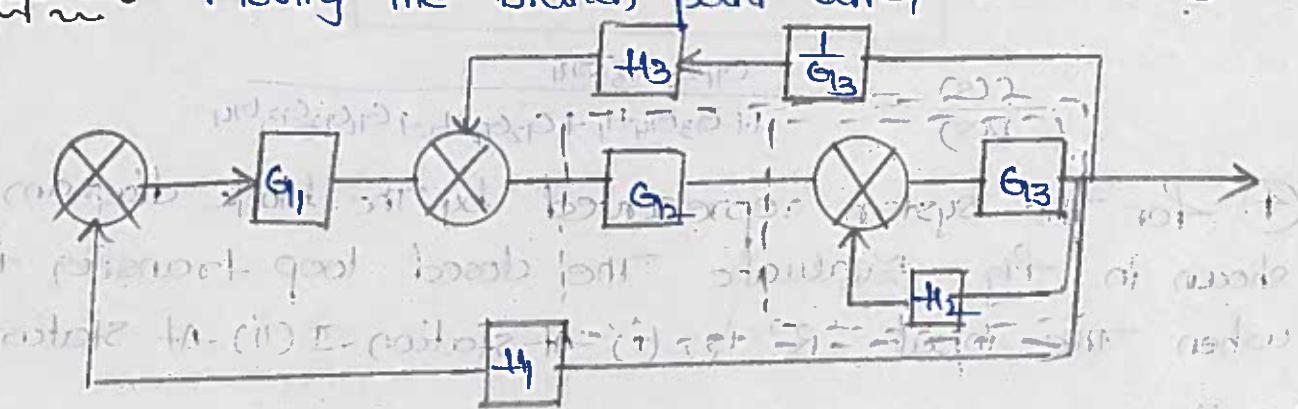
i) Consider, the input $R(s)$ is at station - 1 and, so the input at station - 2 is made zero.

Let the input be $G(s)$.

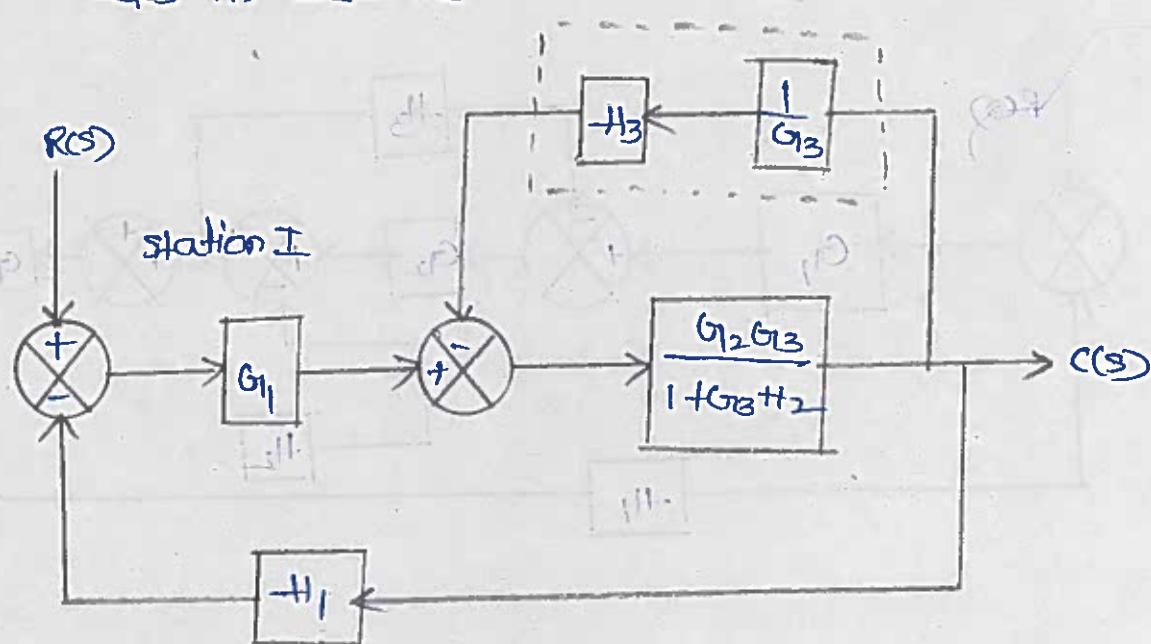
Since, there is no input at station - 2 that summing point can be removed and the resulting block diagram is shown in figure.



Step-1 :- Moving the branch point after the block G_3 .



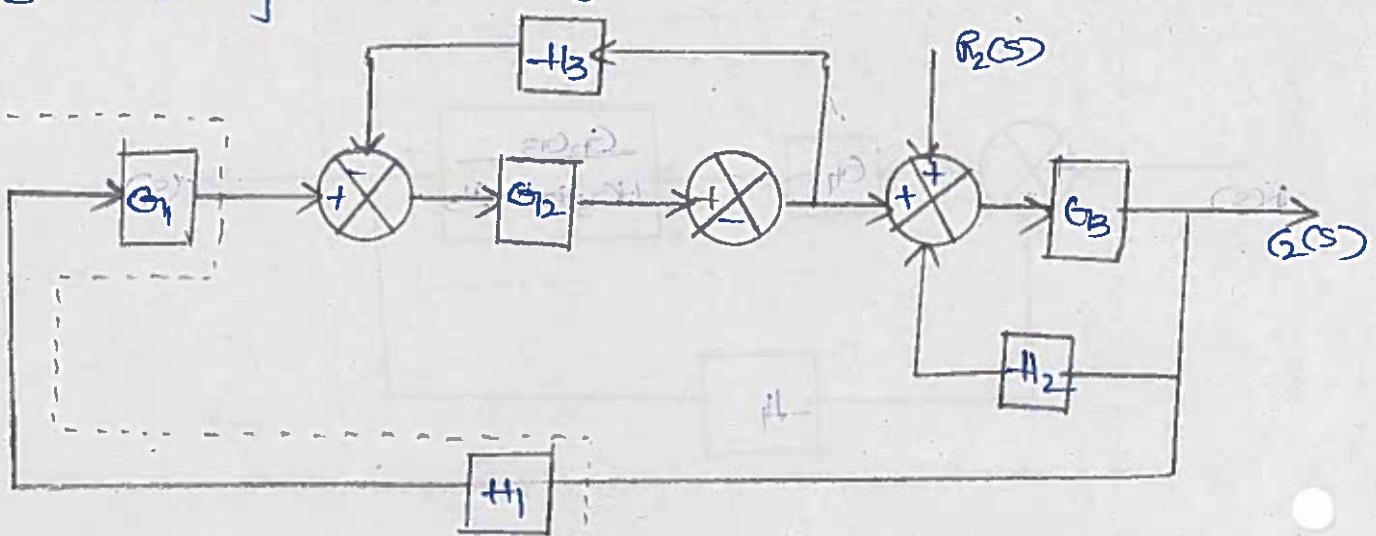
Step-2 :- Elimination of feedback loop and combining the blocks in cascade.



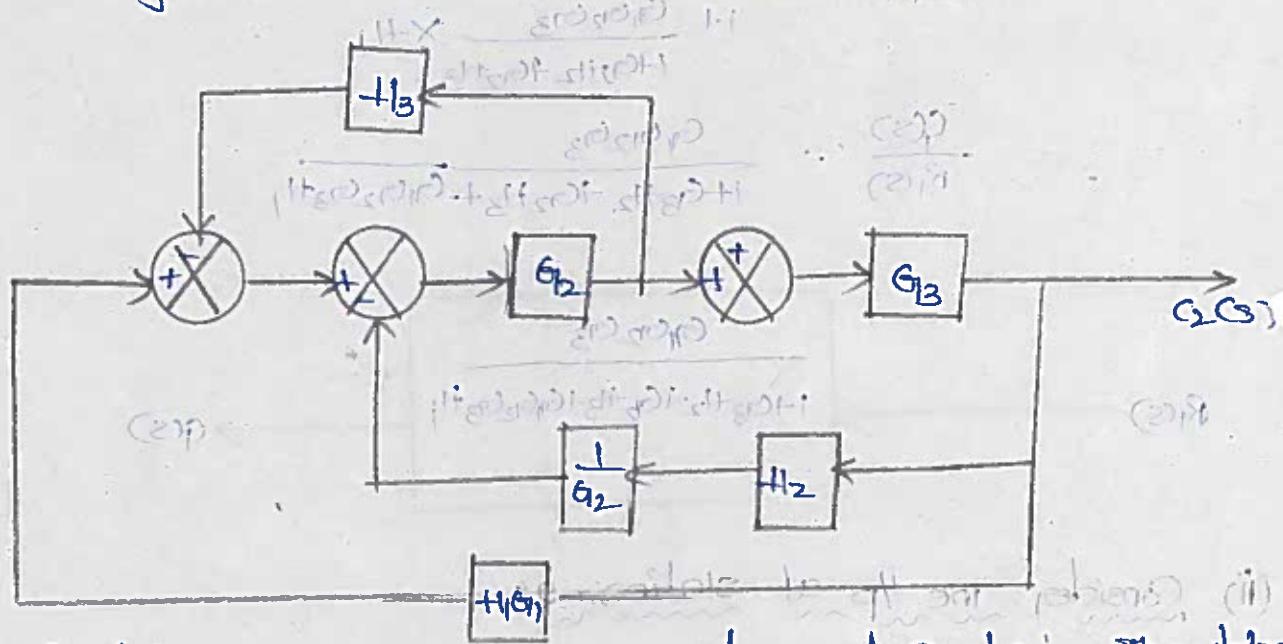
(Consider the $\frac{1}{P} K_2(s)$ at station-II the YP w.r.t. station

-II is. Mode 2010

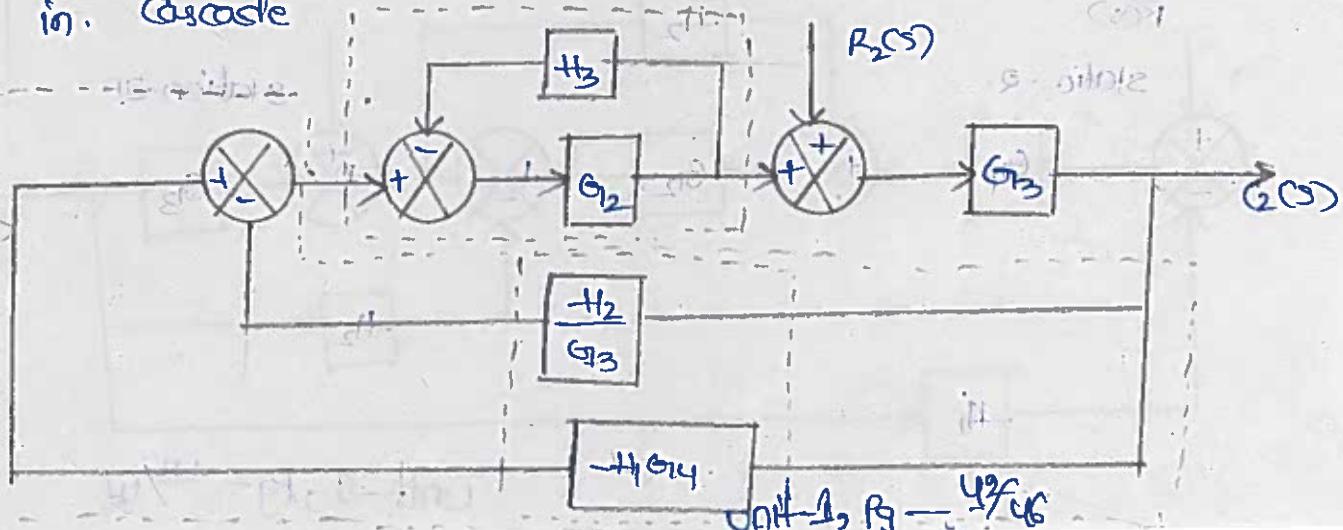
Let the $\frac{1}{P}$ be $G(s)$, since there is no $\frac{1}{P}$ at station-I
That summing point can be removed and -ve sign
can be attached to the feedback path gain H_1 .
The resulting block diagram is shown in figure.



- ① Combining the blocks in cascade and moving the summing point before the block G_2

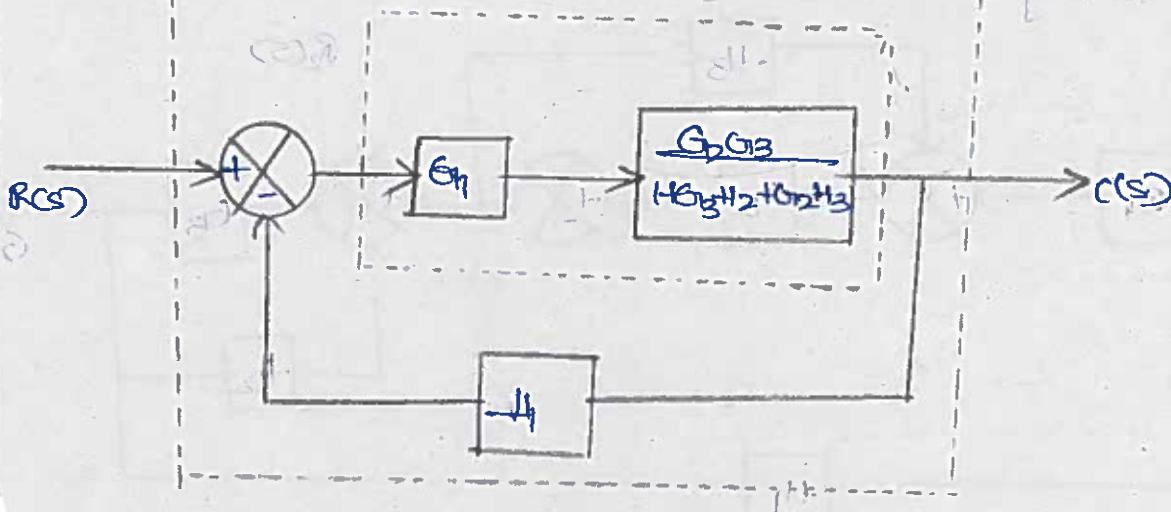


- ② Interchanging the summing points and combining the blocks in cascade



Step-3:- Combining the blocks in cascade and elimination of feedback loop.

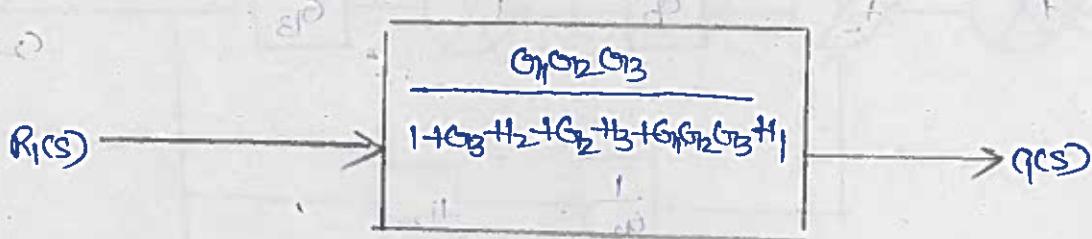
$$\frac{G_1}{1+G_1H_1} = \frac{\frac{G_1 G_2 G_3}{1+G_2 H_2}}{1+G_3 H_3 + \frac{G_1 G_2 G_3}{1+G_2 H_2}} = \frac{G_1 G_2 G_3}{1+G_3 H_3 + G_2 H_2 + G_1 G_2 H_2}$$



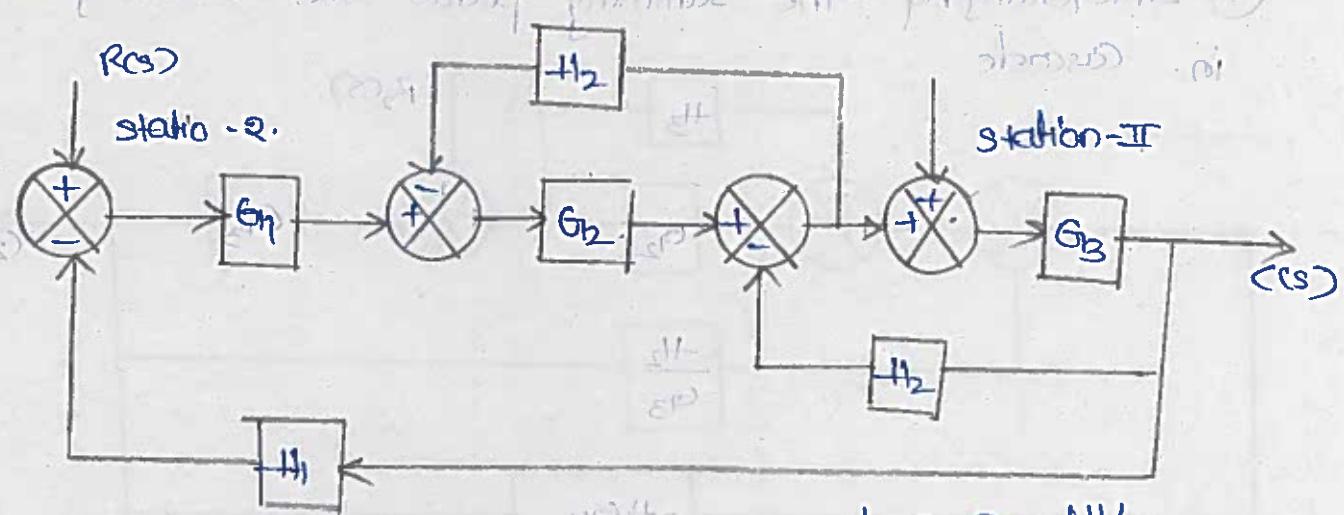
Step-4:- Combining the blocks in cascade and elimination of feedback loop.

$$\frac{G_1}{1+G_1H_1} = \frac{\frac{G_1 G_2 G_3}{1+G_2 H_2 + G_3 H_3}}{1 + \frac{G_1 G_2 G_3}{1+G_2 H_2 + G_3 H_3} \times H_1}$$

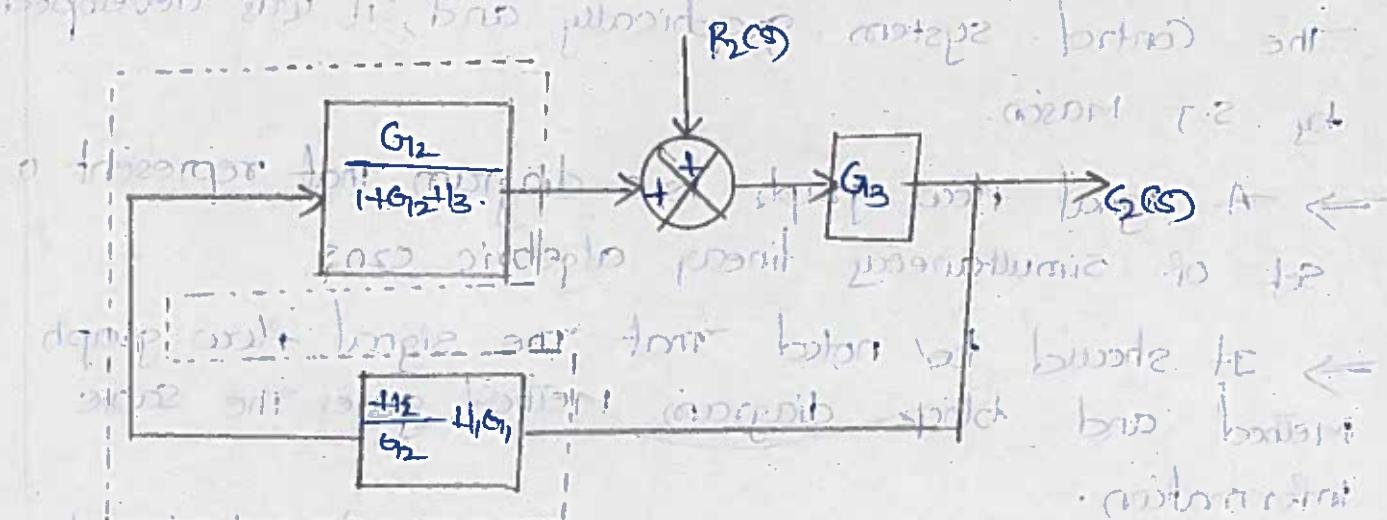
$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3}{1+G_3 H_3 + G_2 H_2 + G_1 G_2 G_3 H_1}$$



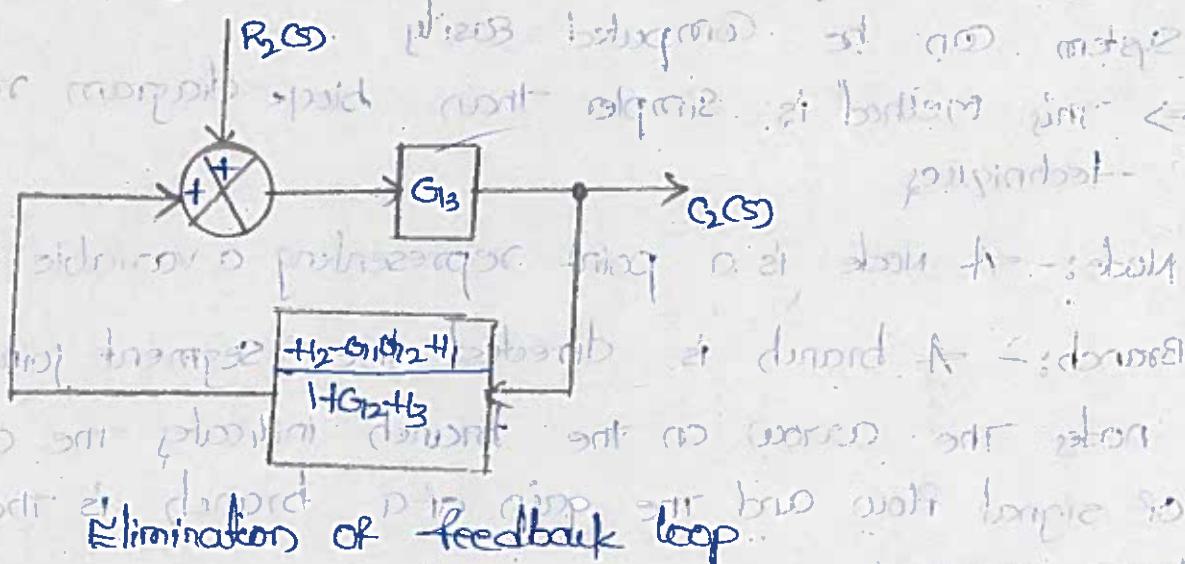
(ii) Consider the I/p at station - 2^a



③ Combining the blocks in parallel and elimination of feedback loop.



④ Combining the blocks in cascade.



$$\frac{G(s)}{R_2(s)} = \frac{G_3}{1 - G_3 \left[\frac{H_2 + G_1 H_2 + H_1}{1 + G_2 + H_3} \right]} = \frac{G_3 [1 + G_2 + H_3]}{1 + G_2 + H_3 - G_3 H_2 + G_1 G_2 G_3 H_1}$$

$$\frac{G(s)}{R_2(s)} = \frac{G_3 [1 + G_2 + H_3]}{1 + G_2 + H_3 - G_3 H_2 + G_1 G_2 G_3 H_1} \rightarrow G(s)$$

To find the value of $G(s)$ we have to eliminate H_1 and H_2 from the equation.

Since $G_1 G_2 G_3$ has already been found, we can substitute it in the equation.

Signal flow Graph:-

The signal flow graph is used to represent the control system graphically and, it was developed by S.T Mason.

- ⇒ A signal flow graph is a diagram that represents a set of simultaneous linear algebraic eqns.
- ⇒ It should be noted that the signal flow graph method and block diagram method give the same information.
- ⇒ The advantages of signal flow graph method is that using "Mason gain formula", the overall gain of the system can be computed easily.
- ⇒ This method is simpler than block diagram reduction techniques.

Node:- A node is a point representing a variable (or) signal

Branch:- A branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow and the gain of a branch is the transmittance.

Transmittance:- The signal when it travels from one node to another is called transmittance.

Transmittance can be real (or) complex.

Input node (or) source:- It is a node that has only outgoing branches.

Output node (or) sink:- It is a node that has only incoming branches.

Open path:- A open path starts at node and ends at another node.

Closed path:- A closed path starts and ends at same node.

Feedback:-

There are two types.

- positive feedback
- Negative feedback

The feedback is a control action in which the output is sampled and a proportional signal is given to input for automatic correction of any changes in desired output.

⇒ why negative feedback is preferred in a closed loop system?

The negative feedback results in better stability in a steady state and reject any disturbance signals.

It also has low sensitivity to parameter variations. Hence, -ve feedback is the preferred in closed loop system.

⇒ what is the effect of positive feedback on stability?

The positive feedback increases the error signal and drives the $\%_p$ to instability. But sometimes the positive F/B. is used in minor loops, in a control system to amplify certain internal signals or parameters.

⇒ what are the %s of Negative feedback?

- * Accuracy in steady state value
- * Rejection of disturbance signals.
- * Low sensitivity to parameter variations.
- * Reduction in gain at the expense of better stability.

Synchos:-

The term Synchro is a generic name for a family of inductive devices, which works on the principle of a rotating transformer [Induction motor]

⇒ The trade names for synchros are setsyn, Autosyn, Telesyn

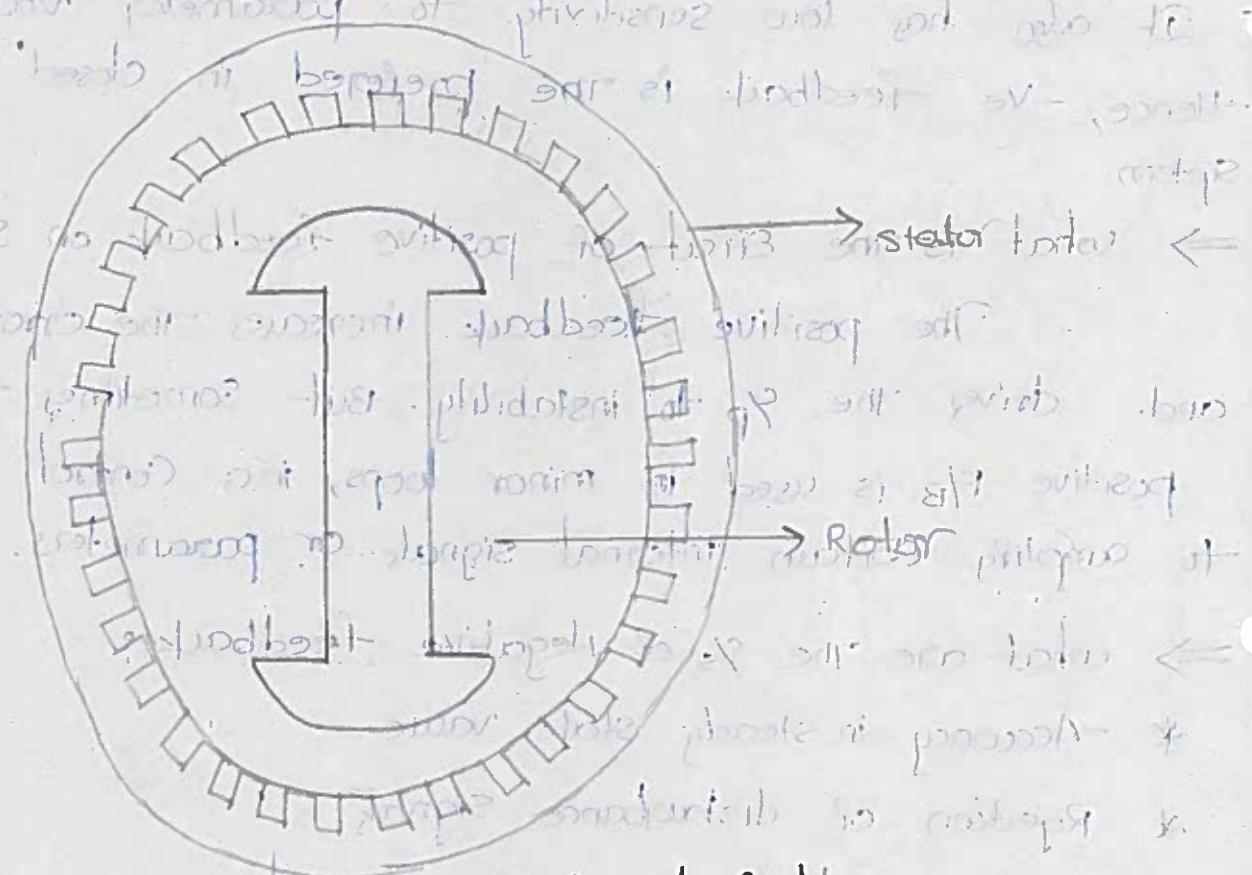
⇒ basically they are electromechanically devices (op) i.e. electromagnetic transducers, which produces an op voltage depending upon angular position of the rotor.

⇒ A Synchro system is performed by interconnection of the devices are called the Synchro transmitter and Synchro receiver [Synchro Control transformer]

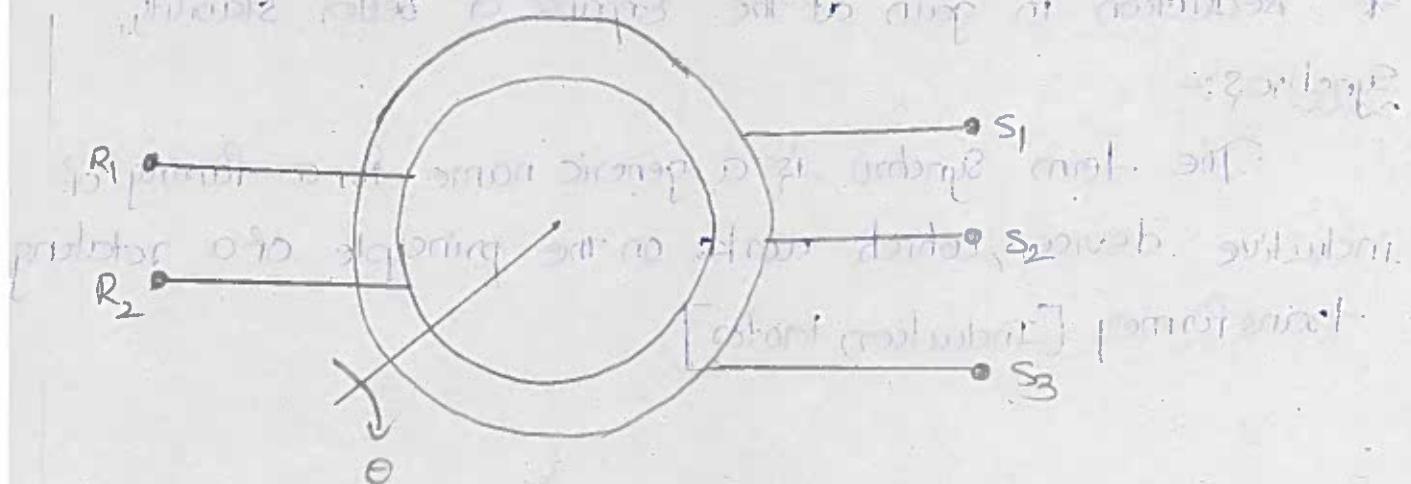
⇒ They are also called synchro pair

⇒ the Synchro pair measure and compare two angular displacements.

Synchro transmitter :-



constructional features :-



Symbol of Synchro transmitter.

UNIT - II

Time Response Analysis

Time Response :- The System is the output of the closed system as a function of time which is denoted by $C(t)$ the time response can be obtained by solving the diff equation governing the System.

The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)+H(s)}$$

standard test signal :-

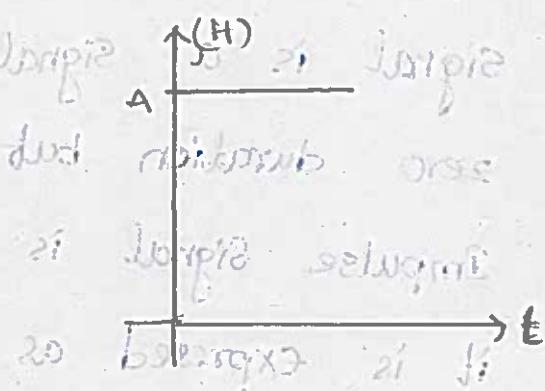
The standard Signals are four types. They are

1. impulse signal
2. step signal
3. Ramp signal
4. parabolic signal

Step Signal :- The step signal are four types. They are. The step signal is a signal whose value changes from "0" to "A" at $t=0$ and remains constant at 'A' for $t>0$ the mathematical representation of Step signal

$$y(t) = A \text{ for } t \geq 0$$

$$y(t) = 0 \text{ for } t < 0$$



~~represent sampling unit~~

Ramp Signal :- The Ramp signal is a signal whose value increases linearly with time from an initial value 'a' at $t=0$. The mathematical representation of Ramp signal is

$$r(t) = At \quad \text{for } t > 0$$

$$r(t) = 0 \quad \text{for } t < 0$$

parabolic Signal :- In parabolic Signal the instantaneous Value varies a Square of the time from an initial value '0' at $t=0$. The mathematical representation of parabolic Signal is

$$r(t) = \frac{At^2}{2} \quad \text{for } t \geq 0$$

$$r(t) = 0 \quad \text{for } t < 0$$

Impulse Signal :- A signal of large magnitude which is available for very short duration is called impulse signal.

Ideal Impulse Signal :- An Ideal impulse signal is a signal with infinite magnitude of zero duration but with an area of "A". The Impulse signal is denoted by $\delta(t)$ & mathematical it is expressed as

$$\delta(t) = \infty, t = 0$$

$$\delta(t) = 0, t \neq 0$$

Order of a System (2) 31

The i/p & output relationship of a control system

can be expressed by n th order differential equation

$$\text{Transfer function } T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

where $P(s)$ = Numerator polynomial

$Q(s)$ = Denominator polynomial

The order of the system is given by max power of s in the denominator polynomial $Q(s)$

$$\text{where } Q(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$$

when $n=0$, the system is zero order system

when $n=1$, the system is 1st order system

when $n=2$, the system is 2nd order system

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_m)}$$

partial fraction expansion :

case 1 :- Function with separate poles.

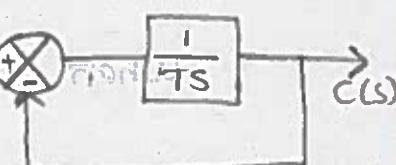
case 2 :- Function with multiple poles.

case 3 :- Function with complex conjugate poles.

Response of 1st order system for unit step input

$$\frac{C(s)}{R(s)} = \frac{1}{Ts}$$

$$R(s)$$



The closed loop transfer function of 1st with unity feedback is shown in figure

$$H \quad \frac{C(s)}{R(s)} = \frac{1/Ts}{1+1/Ts} = \frac{1}{1+Ts} \text{ makes it to zero}$$

Ans: The response of $C(s) = R(s) = \frac{1}{1+Ts}$ if unit step

If the input is unit step, then $R(t) = 1$

$$R(s) = \frac{1}{s} \cdot \frac{(2T)}{s+1/T} = \frac{1}{s} \cdot \frac{2T}{s+2T} = \frac{1}{s} \cdot \frac{2}{s+2} = \frac{2}{s^2 + 2s}$$

By partial fraction $C(s) = \frac{1}{s} - \frac{1}{s+2}$

$$\text{Laplace transform: } C(s) = \frac{1}{s} - \frac{1}{s+2}$$

on taking Laplace transform $(s=0, A=1)$

$$C(s) = \frac{A}{s} + \frac{B}{s+2}$$

$$L[C(s)] = L\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

$$c(t) = 1 - e^{-t/2}$$

$$(in s=0) \quad c(t) = 1 - e^0 = 1 - 1$$

$$(at t=0) \quad c(0) = 0$$

$$\text{when } t=1T, c(t) = 1 - e^{-1/2} = 1 - e^{-1} = 1 - 0.367$$

$$= 0.632$$

$$= \frac{1/T}{s(s+1/T)} = \frac{1/T}{(s+1/T)} = -2$$

$$\frac{1/T}{s(s+1/T)} (s+1/T) = \frac{1/T}{-1/T} = -1$$

$$\left[\frac{1}{s+a} \right] = e^{-at}$$

$$\text{when } t=5T, c(t) = 0.993$$

$$t=5T; c(t) = 1 - e^{-5T/T} = 1 - 0.01 = 0.9817$$

$$\text{when } t=3T, c(t) = 1 - e^{-3T/T} = 1 - e^{-3} = 1 - 0.49 = 0.95$$

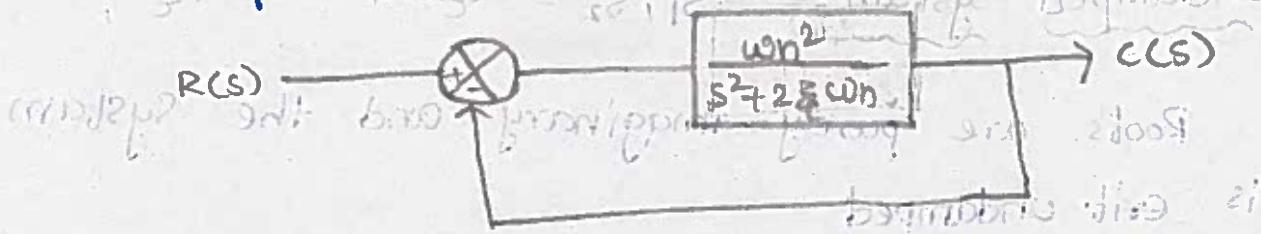
$$c(t) = 1 - e^{-T/T}$$

when $t=0$, $C(t) = 1 - e^0 = 1 - 1 = 0$

when $t=\pi$, $C(t) = 1 - e^{-t/\tau} = 1 - e^{-\pi/\tau} = 1 - 0.367 = 0.632$

when $t=2\pi$, $C(t) = 1 - e^{-2t/\tau} = 1 - e^{-2\pi/\tau} = 1 - 0.135 = 0.865$

The closed loop second order system is shown in figure



The standard form of closed loop transfer function of second order system is given by

where ω_n = undamped natural frequency

ω_0 = Damped

ζ = Damping ratio.

Damping ratio :-

The damping ratio is defined as the ratio of the actual damping to the critical damping. The response $c(t)$ of 2nd Order System depends on the value of the damping ratio.

' ζ ' can be classified as following

Case 1 :- Undamped System, $\zeta = 0$

Case 2 :- Under damped System, $0 < \zeta < 1$

Case 3 :- Critically damped system, $\zeta = 1$.

Case 4 :- Over damped system, $\zeta > 1$

The characteristic equation of Second Order

System

$$s^2 + 2\xi\omega_n s + \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}$$

2.11

$$s_1, s_2 = -\xi\omega_n \pm \sqrt{\omega_n^2(\xi^2 - 1)}$$

$$s_1, s_2 = -\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}$$

Undamped System :- $s_1, s_2 = -\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}$

Root's are purely imaginary and the system is said to be undamped.

Critically damped System :- ($\xi = 1$)

Root's are real and equal and the system is said to be critically damped.

Over damped :- ($\xi < 1$)

$$s_1, s_2 = -\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}$$

Root's are real and unequal and the system is said to be over damped.

$$s_1, s_2 = -\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}$$

$$\begin{aligned} s_1, s_2 &= -\xi\omega_n \pm \omega_n \sqrt{(-1)(1-\xi^2)} \\ &= -\xi\omega_n \pm \omega_n \sqrt{1-\xi^2} \end{aligned}$$

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

Under damped :-

Root's are complex conjugate and the system is said to be under damped.

Response of Undamped Second Order System

for Unit Step Input

The standard form of closed loop transfer function of 2nd order system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

for undamped system:-

$$\zeta = 0 \quad \text{new } \pm j\omega_n \quad \text{new } \pm j\omega_n$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$C(s) = R(s) \frac{\omega_n^2}{s^2 + \omega_n^2}$$

when the input is unit step input

$$C(s) = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

$$s=0, A = \frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2} = 1$$

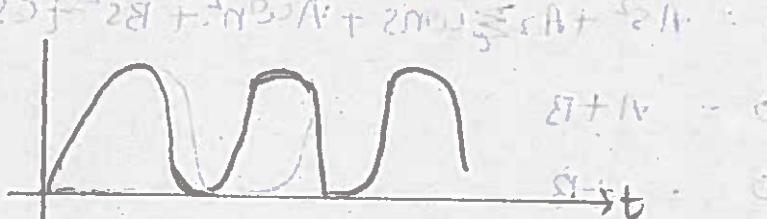
$$\zeta^2 = -\omega_n^2, B = \frac{\omega_n^2 (s^2 + \omega_n^2)}{s(s^2 + \omega_n^2)} = \frac{\omega_n^2}{s} = \frac{\omega_n}{s}$$

$$\zeta = j\omega_n = -s$$

$$s^2 = j^2\omega_n^2, s = j\omega_n$$

$$L^{-1} \left[C(s) = L^{-1} \left[\frac{1}{s} + \frac{s}{s^2 + \omega_n^2} \right] \right]$$

on taking inverse laplace transform above equation.



Response of Under damped Order System

for Unit Step Input

standard form of closed loop transfer function of second order system is given by $\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, Pg-7/ST

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= -\zeta\omega_n \pm j\omega_n\sqrt{(-1)\cdot(1-\zeta^2)}$$

$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n$$

$$\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$$

when the input is unit step is $r(t) = 1, R(s) = 1/s$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = R(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$C(s) = \frac{A}{s} + \frac{Bs+C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s=0, A = \frac{\omega_n^2(s)}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{A(s^2 + 2\zeta\omega_n s + \omega_n^2)}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} + \frac{(Bs+C)s}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$\omega_n^2 = As^2 + A2\zeta\omega_n s + A\omega_n^2 + Bs^2 + Cs$$

$$0 = A + B$$

$$0 = 1 + B$$

$$B = -1$$

$$0 = A2\zeta\omega_n + C$$

$$0 = 2\zeta\omega_n + C$$

$$C = -2\zeta\omega_n$$

$$C(s) = \frac{1}{s} + \frac{-\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \zeta^2 \omega_n^2 - \zeta^2 \omega_n^2 n + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2 (1 - \zeta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta \omega_n + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2} - \frac{\zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2} - \zeta \omega_n \frac{\omega_n}{\omega_n \sqrt{1 - \zeta^2}} \frac{\omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2}$$

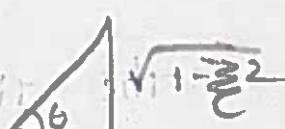
$$C(s) = \frac{1}{s} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2} - \frac{\zeta \omega_n}{\omega_n \sqrt{1 - \zeta^2}} \frac{\omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2}$$

$$c(t) = 1 - e^{-\zeta \omega_n t} \cdot \cos \omega_n t = \frac{\zeta e}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \omega_n t$$

$$c(t) = \frac{1 - e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \left[\sqrt{1 - \zeta^2} \cos \omega_n t - \zeta \sin \omega_n t \right]$$

$$c(t) = \frac{1 - e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \left[\sin \theta \cos \omega_n t - \cos \theta \sin \omega_n t \right]$$

$$\sin \theta = \sqrt{1 - \zeta^2} \quad \cos \theta = \zeta$$



$$c(t) = \frac{1 - e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_n t + \theta)$$

Response of Critically damped Second Order System for unit Step Input?

Standard form of closed loop transfer function of Second order System.

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$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\zeta = 1, \tau(t) = 1 \Rightarrow R(s) = 1/s$$

$$C(s) = R(s) \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\cdot\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

$$C(s) = \frac{A}{s} + \frac{B}{(s + \omega_n)^2} + \frac{C}{s + \omega_n}$$

$$s=0, A = \frac{\omega_n^2 \cdot 0}{0 \cdot (0 + \omega_n)^2} = \frac{\omega_n^2}{0 + \omega_n^2} = 1$$

$$s = -\omega_n, B = \frac{\omega_n^2 \cdot (-\omega_n) \times (0 + \omega_n)^2}{0 \cdot (0 + \omega_n)^2} = \frac{\omega_n^2}{-\omega_n} = -\omega_n$$

$$s = -\omega_n, C = \frac{d}{ds} \left[\frac{\omega_n^2 \times (s + \omega_n)^2}{s + (s + \omega_n)^2} \right]$$

$$C = \frac{d}{ds} \left[\frac{\omega_n^2}{s} \right] = \frac{-\omega_n^2}{s^2} = \frac{-\omega_n^2}{\omega_n^2} = -1$$

On taking Inverse Laplace transfer

Time domain Specifications

The time domain specifications are

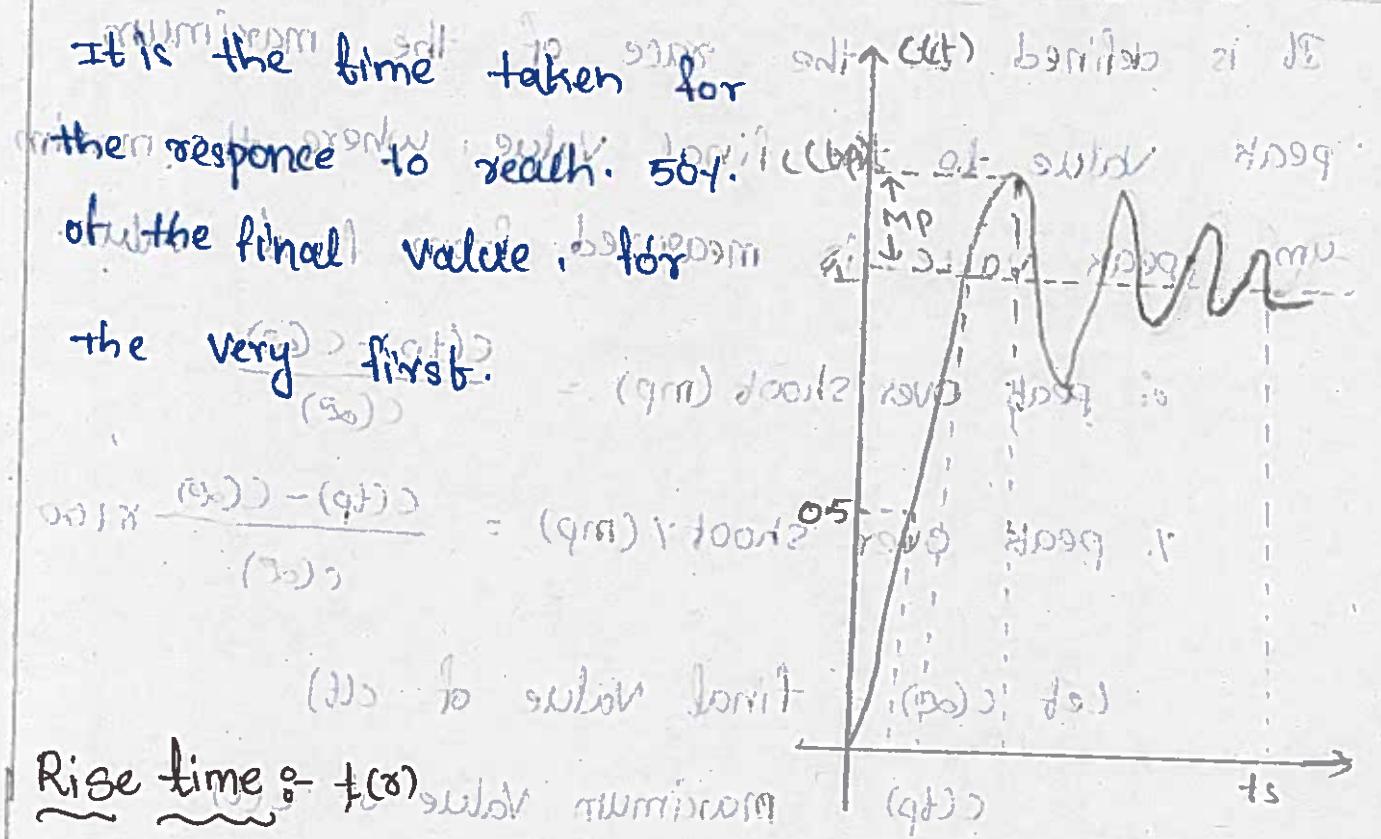
1. Delay time t_d 2. Rise time, t_r

3. peak time, t_p 4. peak-overshoot (or) maximum overshoot

5. settling time t_s .

Delay time & (t_d)

→ (gm) double zero HOSG



Rise time :- t_r is the time taken for the response to rise from 0-100% for the very first time for Under damped system, the rise time is calculated from zero to 100%. but, Over damped system is the time taken by the response to rise from 10-90% for Critically damped system if is the time taken for response to rise from 5-95%.

Peak time :- t_p

It is the time taken for the response to reach the peak value the very first time (or) it is the time taken for the response to reach the peak overshoot M.P.

12 peak over shoot (mp) :-

It is defined as the ratio of the maximum peak value to the final value, where the maximum peak value is measured from final value.

$$\therefore \text{peak over shoot (mp)} = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

$$\% \text{ peak over shoot} \% (\text{mp}) = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

Let $c(\infty)$ = final value of $c(t)$

$c(t_p)$ = maximum value of $c(t)$

Settling time (TS)

It is defined as the time taken by the response to reach and stay within a specified error. It is usually expressed as a percentage of final value. The useful tolerance error is 2% (or) 5% of the final value.

Expressions for time domain specification.

(or) Time domain :

Rise time (t_r)

$$c(t) = 1 + \frac{e^{-\xi \omega t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)$$

The unit step response of Second Order System for under damped case is given as

$$\text{At } t = t_r, c(t) = c(t_r) = 1$$

unit-2, Pg-12/5f

15

The unity feedback system is characterized by

an open loop transfer function $G(s) = \frac{K}{s(s+10)}$ determine the gain K , so that the system will have a damping ratio of 0.5 for the value of K , determine peak overshoot and time at peak for a unit step input.

sds The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{K}{s^2 + 10s + K}$$

standard form of closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = K, \quad \omega_n = \sqrt{K}$$

$$2\zeta\omega_n = 10 \Rightarrow \frac{1}{d.f.} = \frac{1}{\zeta\omega_n} = 5$$

$$\zeta = 0.2$$

$$\% MP = \frac{C(tp) - C(\infty)}{C(\infty)} \times 100$$

$$\left(\frac{C(tp)}{C(\infty)} \right) = e^{-\zeta\pi} / \sqrt{1 - \zeta^2} \times 100$$

$$\% MP = e^{-(0.2)\pi} / \sqrt{1 - (0.2)^2} \times 100$$

$$\% MP = 16.3 \%$$

$$+P = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = 0$$

$$= 0.363$$

Unit-2, Pg- 15/57

② A unity feedback Control system as an

open loop control system. find the rise time, overshoot, peak time and settling time for a step input. Unit's $G(s) = \frac{10}{s(s+2)}$ closed loop transfer function.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{10}{s(s+2)}}{s}$$

$$R(s) = \frac{12}{s}, R(t) = 12$$

Standard form of closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 10, \omega_n = \sqrt{10} = 3.16$$

$$\xi = 0.316$$

$$2\xi\omega_n = 2$$

$$\xi = \frac{1}{\omega_n} = \frac{1}{3.16} = 0.316$$

$$\boxed{\xi = 0.316}$$

$$t_r = \frac{\pi - \theta}{\omega_n}$$

$$\theta = \tan^{-1} \left(\frac{\sqrt{1-\xi^2}}{\xi} \right) = \tan^{-1} \left(\frac{\sqrt{1-(0.316)^2}}{0.316} \right)$$

$$\theta = 71.57^\circ \Rightarrow 1.249 \text{ rad}$$

$$\omega_n = \omega_n \sqrt{1-\xi^2} = 3.16 \sqrt{1-(0.316)^2}$$

$$t_r = \frac{\pi - \theta}{\omega_n} = \frac{\pi - 1.249}{3} \Rightarrow$$

$$\boxed{t_r = 0.63 \text{ sec}}$$

unit 2, Pg - 16/57

$$6 \text{ MP} = \frac{e^{-3\pi/1000}}{\sqrt{1-\zeta^2}} \times 10$$

$$\% \text{ MP} = \frac{e^{-0.316}}{\sqrt{1-0.316^2}} \times 100$$

$$16.07 = 11 \quad \text{initial ref. shift}$$

$$\boxed{\% \text{ MP} = 35.12 \%}$$

initial ref. shift is 0.12

$$t_p = \frac{\pi}{\omega_n}$$

$$t_p = \frac{\pi}{3}$$

$$\boxed{t_p = 1.047 \text{ sec}}$$

settling time constant $T = \frac{1}{\zeta \omega_n} = \frac{1}{0.316 \times 3.16} = 1 \text{ sec}$

57. over shoot, settling time

$$t_s = 3T$$

$$\text{settling time} = 3 \times 1 = 3 \text{ sec}$$

for 2.1. over settling time

$$\text{overshoot} = HT = H \times 1 = 4 \text{ sec}$$

Type Number of a control System:-

The type number is specified by no. of poles of the loop transfer function $G(s)H(s)$. The no. of poles of the loop transfer lying at origin decides the time no. of the system. In general if 'N' no. of poles at the origin then the type no. of 'N' is the loop transfer function can be expressed

as ratio of two polynomial in 's'.
Unit-2, Pg- 17/57

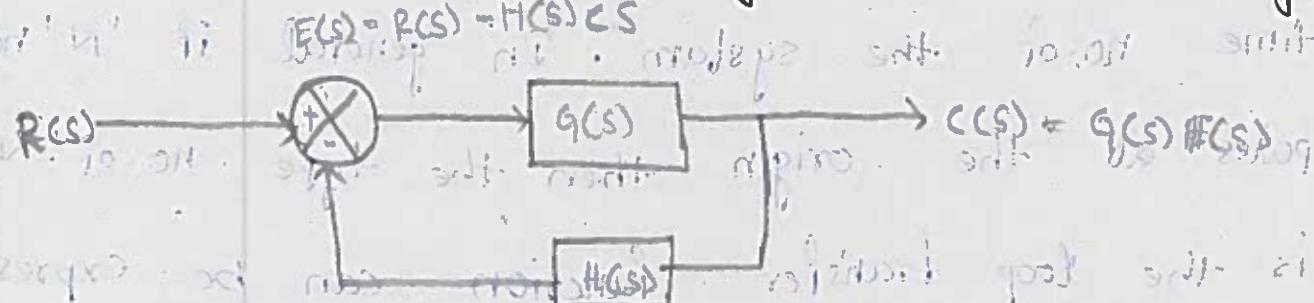
18 where z_1, z_2, z_3 are soon are zeros of the transfer function P_1, P_2, P_3 are soon are poles of transfer function $K = \text{constant}$ $N = \text{no. of poles of the transfer at the origin}$. If $N=0$ than the system type is zero system. If $N=1$ then $N=2$ then the system is type 1 system.

Steady state error

The steady state error is the value of the error signal $e(t)$, when $t \rightarrow \infty$. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of the system & from non-linearity of the system components.

- * The steady state performance of a stable control system is generally decided by steady state error. To step, ramp and parabolic inputs.

Consider a closed loop system is shown in figure



Let $R(s) = \text{i/p signal}$
unit 2, Pg - 18/57

(a) $E(s) = \text{Error signal}$

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$H(s)C(s) = \text{feed-back signal}$

$C(s) = \text{output signal or response}$

$E(s) = R(s) - H(s)C(s) \quad \text{--- (1)}$

$C(s) = G(s)E(s) \quad \text{--- (2)}$

$E(s) = R(s) - H(s)G(s)E(s)$

$$E(s) = [1 + G(s)H(s)]^{-1}R(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

Let $E(t) = \text{Error signal in time domain}$

$$e(t) = L^{-1}[E(s)] = L^{-1}\left[\frac{R(s)}{1 + G(s)H(s)}\right]$$

Let $e_{ss} = \text{steady state error}$

$$e_{ss} = L^{-1}[A(t)] \text{ then if } A(t) = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

The steady state error is defined as the value

$e(t)$ then $t \rightarrow \infty$ $e_{ss} = \lim_{t \rightarrow \infty} e(t)$

the final value theorem of L.T states that

$$\text{then if } A(t) = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

Using final value theorem, the steady state error

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} E(s)$$

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} \frac{s}{s + G(s) + H(s)}$$

static error Constants :-

When a control system is excited with standard input signal, the steady state error may be '0' (or) constant (or) error. The value of steady state error depends on the type number and the input signal.

Type - 0 :- System will have a constant steady state error when the input is step signal

Type - 1 :- System will have a constant steady state error when the input is ramp signal (or) velocity signal

Type - 2 :- System will have a constant steady state error when the input is parabolic signal (or) acceleration signal

positional error constant ζ (K_p)

$$\text{positional error constant } K_p = \lim_{s \rightarrow 0} G(s) H(s)$$

velocity error constant (K_v) :-

$$K_v = \lim_{s \rightarrow 0} s G(s) H(s)$$

acceleration error constant (K_a)

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) H(s)$$

CONCEPT OF STABILITY AND ROOT LOCUS

Definition's of stability :-

Definition - 1 :-

1. A system is stable if its output is bounded finite for any bounded input.

Definition - 2 :-

2. A system is asymptotically stable; if in the absence of the input, the output tends to zero irrespective of initial condition

3. A system is unstable if for a bounded input signal the output is of infinite amplitude or oscillatory

4. If a system output is stable for a limited range of variation's of its parameter's, then the system is called conditionally stable system.

Routh-Hurwitz criterion :-

Case(i) Normal Routh array

Using Routh or criterion, determine the stability of the system represented by the characteristic equation $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$ Comment on the location of the roots on the characteristic equation.

The given characteristic equation is
unit-2, Pg-22/5t

$$s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$$

(23) 23

The given characteristic equation is fourth order eqn and it has four roots.

since, the highest power of s is given number the first row is formed by coefficients of even powers of s.

and second order row is formed by coefficients of odd power of s.

$$\begin{array}{r} s^4 : 1 \quad 18 \quad 5 \\ - \quad - \quad - \quad - \\ s^3 : 8 \quad 16 \quad 1 \\ - \quad - \quad - \quad - \end{array}$$

s^3 row is devived by 8!

$$s^4 : 1 \quad 18 \quad 5$$

$$s^3 : 1 \quad 2$$

$$s^2 : 16 \quad 5$$

$$s^1 : 1.68$$

$$s^0 : 5$$

$$(-1) \frac{\begin{vmatrix} 1 & 18 \\ 1 & 2 \end{vmatrix}}{1} = \frac{(-1)[2-18]}{1} = 16$$

$$(-1) \frac{\begin{vmatrix} 1 & 5 \\ 8 & 0 \end{vmatrix}}{1} = \frac{(-1)(0-5)}{1} = 5$$

$$(-1) \frac{\begin{vmatrix} 1 & 2 \\ 16 & 5 \end{vmatrix}}{16} = \frac{(-1) \frac{5-32}{16}}{16} = -1.68$$

$$(-1) \frac{\begin{vmatrix} 16 & 5 \\ 16 & 0 \end{vmatrix}}{1.68} = \frac{(-1)[0-5(1.68)]}{1.68} = 5$$

1. Given system is stable there is no changes in the sign in first column of the Routh array

2. All the 4 root's are lying on left side of s-plane.

24 Case ii :- A Row of All zeros.

Construct Routh array determine the stability of the system whose characteristic equation is, $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$ also determine the no. of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

Solutions Given characteristic equation is

$$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

The given characteristic equation is 6th order equation so it has 6 roots

since the highest power of s is even number the first row is formed by coefficients of even power of s and second row is formed by coefficients of odd powers of s.

$$\begin{array}{cccc} s^6 : & 1 & 8 & 20 \\ s^5 : & - & - & - \\ \hline 1 & 2 & 12 & 16 \end{array}$$

s^5 is devied by 2

$$\begin{array}{cccc} s^6 : & 1 & 8 & 20 \\ s^5 : & - & - & - \\ \hline 1 & 0 & 12 & 16 \end{array}$$

$$s^5 : 1 \quad 6 \quad 8$$

$$s^4 : 2 \quad 12 \quad 16$$

$$s^3 : 0 \quad 0 \quad 0$$

$$s^2 : 1 \quad 3$$

$$s^1 : 6 \quad 16$$

$$s^0 : 0 \quad 3$$

$$s^0 : 16$$

$$(-1) \frac{\begin{vmatrix} 1 & 8 \\ 1 & 6 \end{vmatrix}}{1} = \frac{(-1)(6-8)}{1} = 2$$

$$(-1) \frac{\begin{vmatrix} 1 & 20 \\ 1 & 8 \end{vmatrix}}{1} = \frac{(-1)(8-20)}{1} = 12$$

$$(-1) \frac{\begin{vmatrix} 1 & 16 \\ 1 & 0 \end{vmatrix}}{1} = \frac{(-1)(0-16)}{1} = 16$$

$$(-1) \frac{\begin{vmatrix} 1 & 6 \\ 2 & 12 \end{vmatrix}}{2} = \frac{(-1)(12-12)}{2} = 0$$

$$(-1) \frac{\begin{vmatrix} 1 & 8 \\ 2 & 16 \end{vmatrix}}{2} = \frac{(-1)(16-16)}{2} = 0$$

$$A(s) = 2s^4 + 12s^3 + 16s^2 + 16s + 16$$

$$\frac{dA(s)}{ds} = 8s^3 + 36s^2 + 32s + 16$$

unit-2, Pg - 24/57

$$\begin{array}{l}
 \text{minimum } A(s) = 2s^4 + 12s^3 + 16s^2 \\
 \text{for } s^2 = 2 \Rightarrow s = \pm\sqrt{2} \\
 \text{minimum } f_0 = 2x^2 + 12x + 16 = 0 \\
 x_1 x_2 = -2, -14 \\
 s = \pm\sqrt{x_1}, \quad s = \pm\sqrt{x_2} \\
 s = \pm\sqrt{-2} = \pm\sqrt{-4} \\
 s = \pm\sqrt{2}
 \end{array}
 \quad
 \begin{array}{l}
 \frac{d(A)}{ds} = s^3 + 3s^2 \\
 \text{at } s = 0 \\
 (-1) \begin{vmatrix} 2 & 12 \\ 1 & 3 \end{vmatrix} = \frac{(-1)(6-12)}{0.33} = 6 \\
 (-1) \begin{vmatrix} 2 & 16 \\ 1 & 6 \end{vmatrix} = \frac{(-1)(0-16)}{1} = 16 \\
 (-1) \begin{vmatrix} 6 & 16 \\ 0.33 & 6 \end{vmatrix} = \frac{(-1)(0-16(0.3))}{0.33} = 16
 \end{array}$$

The system is marginally stable system. (or)

Limited stable system system. four root's are lying on Imaginary axis, and two root's are lying on left half of the s-plane.

Case (ii)

First element of Row is zero but other element s are not zero

Construct Routh array and determine the stability of the system represented by the characteristic equation $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$ Comment on the location of the Root's of characteristic equation.

Solution:-

Given characteristic equation

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

The given characteristic equation is 5th order equation and so it has 5 roots. Unit-2 Pg-25/57

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since the highest power of 's' is odd number
 the first row is formed by the coefficients of odd powers of 's' and second row is formed by coefficient of even powers of 's'.

$$S^5 : \begin{matrix} 1 & 2 & 3 \end{matrix}$$

$$S^4 : \begin{matrix} 1 & 2 & 5 \end{matrix}$$

$$S^3 : \begin{matrix} 0 & -2 \end{matrix}$$

0 is Replaced by positive

constant:

$$S^3 : \epsilon - 2$$

$$S^2 : 0 \rightarrow \epsilon$$

$$S^1 : \frac{2(\epsilon t)}{\epsilon} 5$$

$$S^0 : \frac{(-5\epsilon^2 + 4\epsilon + 4)}{2\epsilon + 2}$$

$$S^0 : 5$$

$$(-1) \cdot \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = \frac{(-1)(2-2)}{1} = 0$$

$$(-1) \cdot \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = \frac{(-1)(5-3)}{1} = -2$$

$$(-1) \cdot \begin{vmatrix} 1 & 2 \\ \epsilon & -2 \end{vmatrix} = \frac{(-1)(-2-2\epsilon)}{\epsilon} = \frac{-2(1+\epsilon)}{\epsilon}$$

$$(-1) \cdot \begin{vmatrix} 1 & 5 \\ \epsilon & 0 \end{vmatrix} = \frac{(-1)(0-5\epsilon)}{\epsilon} = 5$$

$$(-1) \cdot \begin{vmatrix} \epsilon & -2 \\ 2(\epsilon t) & 5 \end{vmatrix} = \frac{(-1)[5\epsilon - 2 \cdot \frac{2(\epsilon t)}{\epsilon}]}{2(\epsilon t)} = \frac{-5\epsilon^2 + 4\epsilon + 4}{2\epsilon + 2}$$

$$= \frac{(-5\epsilon^2 + 4\epsilon + 4)}{2\epsilon + 2}$$

Again t is replaced by 0

$$S^5 : \begin{matrix} 1 & 2 & 3 \end{matrix}$$

$$S^4 : \begin{matrix} 1 & 2 & 5 \end{matrix}$$

$$S^3 : \begin{matrix} 0 & -2 \end{matrix}$$

$$S^2 : \begin{matrix} 2\epsilon & 5 \end{matrix}$$

$$S^1 : -2$$

$$S^0 : 5$$

$$(-1) \cdot \begin{vmatrix} \frac{2(\epsilon t)}{\epsilon} & 5 \\ -5\epsilon^2 + 4\epsilon + 4 & 0 \end{vmatrix} = 5$$

$$\frac{-(5\epsilon^2 + 4\epsilon + 4)}{2\epsilon + 2}$$

(i) Given System is unstable.

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(27) 27
 Q) Two Roots are lying on Right half of s-plane
 and remaining three Roots are lying on Left half of s-plane.

- * By Routh stability criterion determine the stability of the system represented by the characteristic equation $9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$ comment on the location of roots of characteristic equation.

Solution: Given characteristic equation

$$9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$$

The given characteristic equation is 5th order equation and so it has 5 roots.

The highest power of 's' is odd number, the first row is formed by the coefficient of odd powers of 's' and second row is formed by the even powers.

$$s^5: \begin{matrix} 9 & 10 & 9 \end{matrix}$$

$$s^4: \begin{matrix} -20 & -1 & -10 \end{matrix}$$

$$s^3: \begin{matrix} 9.5 & 4.5 \end{matrix}$$

$$s^2: \begin{matrix} 8.4 \end{matrix}$$

$$\frac{\begin{vmatrix} 9 & 10 \\ -20 & -1 \end{vmatrix}}{-20} = \frac{(-1)(90+200)}{-20} = 9.5$$

$$\frac{\begin{vmatrix} 9 & -9 \\ -20 & -10 \end{vmatrix}}{-20} = \frac{(-1)(-90-180)}{-20} = 4.5$$

$$\frac{\begin{vmatrix} -20 & -1 \\ 9.5 & 4.5 \end{vmatrix}}{9.5} = \frac{(-20)(4.5)-95}{9.5} = 8.4$$

$$\frac{\begin{vmatrix} -20 & -10 \\ 9.5 & 0 \end{vmatrix}}{9.5} = \frac{(-1)(0+10(9.5))}{9.5}$$

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$$s^5 : \begin{matrix} 9 & 10 & 9 \end{matrix}$$

$$s^4 : \begin{matrix} -20 & -1 & -10 \end{matrix}$$

$$s^3 : \begin{matrix} 9.5 & -13.5 \end{matrix}$$

$$s^2 : \begin{matrix} -29.27 & -10 \end{matrix}$$

$$s^1 : \begin{matrix} -16.74 \end{matrix}$$

$$s^0 : \begin{matrix} -10 \end{matrix}$$

$$(E) \begin{array}{c} | 9 & -10 \\ -20 & -1 \\ \hline -20 \end{array} = 9.5$$

$$(E) \begin{array}{c} | 9 & -9 \\ -20 & -10 \\ \hline -20 \end{array} = \frac{(E) (-90 - 180)}{-20} = -13.5$$

$$(E) \begin{array}{c} | -20 & -1 \\ 9.5 & -13.5 \\ \hline 9.5 \end{array} = -29.27$$

$$(E) \begin{array}{c} | -20 & -10 \\ 9.5 & 0 \\ \hline 9.5 \end{array} = -16.74$$

$$(E) \begin{array}{c} | -29.27 & -10 \\ -16.74 & 0 \\ \hline -16.74 \end{array} = -10$$

1. Given system is unstable
 2. three root's are lying on right half of s-plane and two root's are lying on left half of s-plane.

③. The characteristic polynomial of a system is
 $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 23s^2 + 23s + 15 = 0$ determine the location of the root's on s-plane and hence the stability of the system.

Solution: Given characteristic equation is

$$s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 23s^2 + 23s + 15 = 0$$

The given characteristic equation is 7th order equation and so, it has 7 roots.

The highest power of 's' is odd number. the ⁽¹⁾ (29)
 first row is formed by the coefficient of odd powers
 of 's' and second row is formed by even power of
 's'

$$s^7 : 1 \ 24 \ 24 \ 23$$

$$\begin{array}{cccc} s^6 : & 9 & 24 & 24 \\ \hline & - & - & - & 15 \end{array}$$

s^6 row is divided by 3

$$s^7 : 1 \ 24 \ 24 \ 23$$

$$s^6 : 3 \ 8 \ 8 \ 5$$

$$\begin{array}{cccc} s^5 : & 21.33 & 21.33 & 21.33 \\ \hline & - & - & - & - \end{array}$$

s^5 row is divided by 21.33

$$s^6 : 3 \ 8 \ 8 \ 5$$

$$s^5 : 1 \ 1 \ 1$$

$$\begin{array}{cccc} s^4 : & 5 & 5 & 5 \\ \hline & - & - & - & - \end{array}$$

s^4 row is divided by 5

$$s^5 : 1 \ 1 \ 1$$

$$s^4 : 1 \ 1 \ 1$$

$$\begin{array}{cccc} s^3 : & 0 & 0 & 0 \\ \hline & - & - & - & - \end{array}$$

$$A(s) = s^4 + s^2 + 1$$

$$\frac{d}{ds} A(s) = 4s^3 + 2s$$

$$s^3 : 4 \ 2$$

$$(-1) \cdot \frac{1 \ 24}{3 \ 8} = \frac{(-1)(8-72)}{3} = 24$$

$$(1) \frac{1 \ 23}{3 \ 5} = \frac{(1)(5-3 \times 23)}{3} = 21.33$$

$$21.33$$

$$(-1) \frac{3 \ 8}{1 \ 1} = \frac{(-1)(3-8)}{1} = 5$$

$$(-1) \frac{3 \ 5}{1 \ 0} = \frac{(-1)(0-5)}{1} = 5$$

$$(1) \frac{1 \ 1}{4 \ 2} = \frac{(1)(2-4)}{4} = 0.5$$

$$(-1) \frac{1 \ 1}{4 \ 0} = \frac{(-1)(0-4)}{4} = 1$$

$$(-1) \frac{4 \ 2}{0.5 \ 1} = \frac{(-1)[4(0)-1(0.5)]}{0.5} = -6$$

$$(-1) \frac{4 \ 1}{0.5 \ 0} = \frac{(-1)[4(0)-1(0.5)]}{0.5}$$

$$s^2 = 0.5 + j$$

$$s^1 = -6$$

$$s^2 = 1$$

$$s^4 + s^2 + 1 = 0$$

$$s^2 = x^2$$

$$x^2 + x + 1 = 0$$

$$x_1 x_2 = -\frac{1}{2} \pm j \frac{\sqrt{3}}{2}$$

$$= 0.5 \pm j 0.866$$

$$s_1 s_2 = \pm \sqrt{x_1}$$

$$s_1 s_2 = \pm \sqrt{0.5 + j 0.866}$$

$$s = \pm \sqrt{1 + j 120^\circ}$$

$$s_1 s_2 = \pm (0.5 \pm j 0.866)$$

$$s_3 s_4 = \pm \sqrt{x_2}$$

$$= \pm \sqrt{-0.5 - j 0.866}$$

$$= \pm \sqrt{1 + j 120^\circ}$$

$$= \pm 1 \cdot j 120^\circ$$

$$s_3 s_4 = \pm (0.5 \pm j 0.866)$$

1. Given system is unstable

2. two Roots are lying on Right half of S-plane and remaining five Roots are left half of S-plane.

* Use the Routh stability criterion to determine the location of roots on s-plane and hence the stability for the system represented by characteristic equation. $s^5 + us^4 + 8s^3 + 8s^2 + 7s + u = 0$

The characteristic equation is

$$s^5 + us^4 + 8s^3 + 8s^2 + 7s + u = 0$$

The given characteristic equation is 5th order equation and so it has 5 roots.

The highest power of 's' is odd number. The first row is formed by the coefficients of odd power of 's' and second row is formed by the coefficients of even powers of 's'.

$$\begin{array}{r} s^5 : 1 \ 8 \ 7 \\ s^4 : 4 \ 8 \ 4 \\ \hline - \quad - \quad - \end{array}$$

s^4 row is divided by u

$$\begin{array}{r} s^5 : 1 \ 8 \ 7 \\ s^4 : 1 \ 2 \ 1 \\ \hline s^3 : 6 \ 6 \ 1 \\ \hline - \quad - \quad - \end{array}$$

s^3 row is divided by u

$$s^4 : 1 \ 2 \ 1$$

$$s^3 : 1 \ 1$$

$$s^2 : 1 \ 1$$

$$(-1) \frac{\begin{vmatrix} 1 & 8 \\ 4 & 2 \end{vmatrix}}{1} = \frac{(-1)(2-8)}{1} = 6$$

$$(-1) \frac{\begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix}}{1} = \frac{(-1)(1-7)}{1} = 6$$

$$(-1) \frac{\begin{vmatrix} 1 & -2 \\ 6 & 1 \end{vmatrix}}{1} = \frac{(-1)(1-2)}{1} = 1$$

$$(-1) \frac{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}}{1} = \frac{(-1)(0-1)}{1} = 1$$

$$(-1) \frac{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}{1} = \frac{(-1)(1-1)}{1} = 0$$

$$(3+2)(1+2)$$

$$+ (3+2)(1+2)$$

$$(3+2)(1+2)$$

$s^2 \cdot \text{row}$ is divided by 1

$$s^3 : 1 \quad 0 = s^3 + 2s^2 + 2s + K$$

$$s^1 : 0$$

$$s^0 : 1$$

$$0 = s^3 + 2s^2 + 2s + K$$

1. Given system is marginally stable system

2. All root's are lying on left half of s-plane

$$= 0 =$$

minimum locus of 2 to right side

* Determine the range of K for stability of unity feed back system whose open loop transfer function

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Solutions

Given that

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

$$H(s) = 1$$

the closed loop transfer function.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1 + \frac{K}{s(s+1)(s+2)}}$$

$$= \frac{\frac{K}{s(s+1)(s+2)}}{\frac{s(s+1)(s+2) + K}{s(s+1)(s+2)}}$$

of this point the total no. of real poles and zeros are 1 which is an odd number hence the entire negative real axis will be a part of root locus.

Step -3 To Find angles of Asymptotes and centroid.

since there are three poles the no. of root locus Branches are 3 there is no infinite zero Hence all the three root Branches end at zero at infinity.

$$\text{Angles of Asymptotes} = \pm \frac{180^\circ(2q+1)}{n-m}$$

when $q=0$

$$\text{Angles of Asymptotes} = \pm \frac{180^\circ[2(0)+1]}{3-0} = \pm 60^\circ$$

when $q=1$

$$\text{Angles of Asymptotes} = \pm \frac{180^\circ[2(1)+1]}{3-0} = \pm 180^\circ$$

when $q=2$

$$\text{Angles of Asymptotes} = \pm \frac{180^\circ[2(2)+1]}{3-0} = \pm 60^\circ = \pm 420^\circ$$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{\text{Total no. of poles} - \text{Total no. of zeros}}$$

$$36 \quad \text{Given } G(s) = \frac{1 + (2 + j3)}{s - 3} + \frac{1 - (2 - j3)}{s - 0} \quad (36)$$

$= -\frac{4}{3}$

Centroid = -1.33

Step 4 To find the Break-away and Break-in points.

The closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{K / s(s^2 + 4s + 13)}{1 + \frac{K}{s(s^2 + 4s + 13)}}$$

$$= \frac{K}{s(s^2 + 4s + 13) + K}$$

$$\frac{C(s)}{R(s)} = \frac{K}{s^3 + 4s^2 + 13s + K}$$

The characteristic equation is $s^3 + 4s^2 + 13s + K = 0$

$$K = -(s^3 + 4s^2 + 13s)$$

$$\frac{dK}{ds} = -(3s^2 + 8s + 13)$$

$$0 = -(3s^2 + 8s + 13)$$

$$3s^2 + 8s + 13 = 0$$

$$s_1 s_2 = \frac{-8 \pm \sqrt{64 - 4(3)(13)}}{2(3)}$$

$$\frac{-8 \pm \sqrt{-92}}{6}$$

$$s_1, s_2 = -1.33 \pm j1.6$$

when

$$s = -1.33 + j1.6$$

The value of 'K' is given by $K = -[(-1.33 + j1.6)^3 +$

$$4(-1.33 + j1.6)^2 + 13(-1.33 + j1.6)]$$

$K \neq$ positive and real

when

$$s = -1.33 - j1.6$$

the value of 'K' is given by $K = -[(-1.33 - j1.6)$

$$+ 4(-1.33 - j1.6)^2 + 13(-1.33 - j1.6)]$$

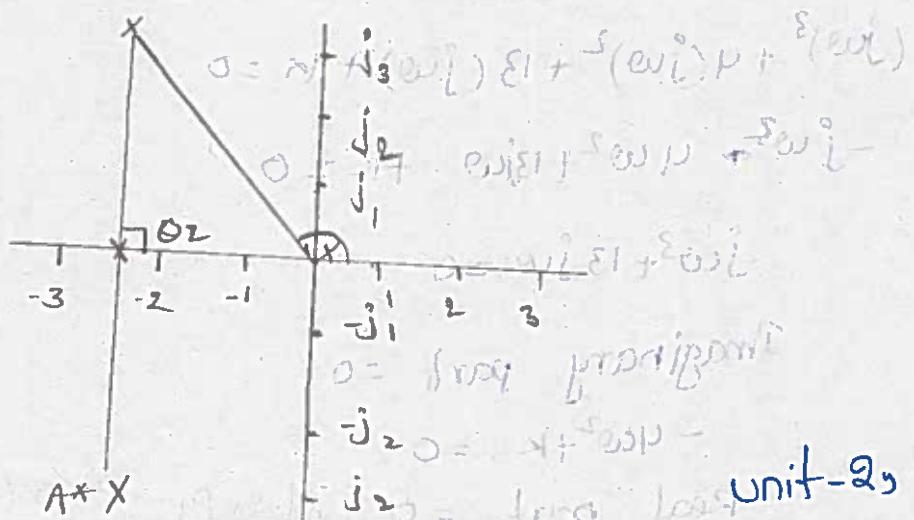
$K \neq$ positive and real

since the value of 'K' for $s = -1.33 \pm j1.6$ are not positive and real, these points are not an actual Break away (or) Break in points the Root's locus has neither Break away nor Break in points.

Step 5

To Find Angle of Departure and

Angle of Arrival:



$$\theta = 180^\circ - \tan^{-1}(3/2) = 123.69$$

$$\phi = 90^\circ$$

180° (sum of angles of vectors to
Angle of Departure = the Complex pole A from other
poles) + sum of angles of vectors
to the Complex pole A
from zeros

$$\begin{aligned}\text{Angle of Departure} &= 180^\circ - (\phi_1 + \phi_2) + (\theta) \\ &= 180^\circ - (123.69^\circ + 90^\circ)\end{aligned}$$

$$\text{Angle of Departure} = -33.69 \approx -33.7^\circ$$

Angle of departure from complex pole A^*

$$\text{AD.A}^* = 33.7^\circ$$

Step - 6

To find the crossing point on Imaginary

the characteristic equation is given by

$$s^3 + 4s^2 + 13s + k = 0$$

putting s is replaced by $-j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + k = 0$$

$$-j\omega^3 + 4\omega^2 + 13j\omega + k = 0$$

$$-j\omega^3 + 13j\omega = 0$$

Imaginary part = 0

$$-4\omega^2 + k = 0$$

Real part = 0 unit-2, Pg - 38/54

$$-j\omega^3 + 13j\omega = 0$$

$$\omega^2 = 13\omega$$

$$-4\omega^2 + k = 0$$

$$-4 \times 13 + k = 0 \quad \text{so } k = 52$$

$$k = 52$$

$$s = -2 \omega j/2 \text{ also } \omega^2 = 13 \text{ so } \omega = \sqrt{13} \text{ rad/sec}$$

$$\omega = \sqrt{13} \text{ rad/sec}$$

$$\omega = \pm 3.6$$

Draw the Root Locus of the system whose open loop Transfer function $G(s) = \frac{k}{s(s+2)(s+4)}$ Find the value of 'k' so that the Damping Ratio of the closed loop system is 0.5.

Given that

$$G(s) = \frac{k}{s(s+2)(s+4)}$$

Step - 1

To locate poles and zeros.

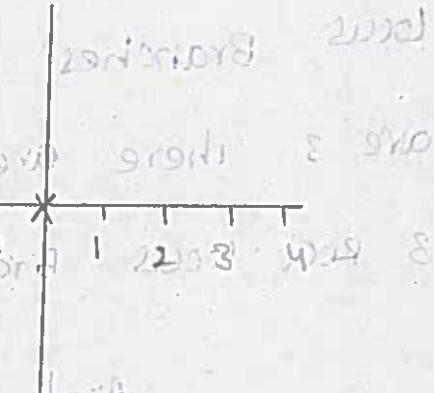
The poles of open loop Transfer function are the roots of the equation.

$$s(s+2)(s+4) = 0$$

$$P_1 = s_1 = 0$$

$$P_2 = s_2 = -2$$

$$P_3 = s_3 = -4$$



Step - 2

To find the Root's locas on Real axis

There are three pole's on the Real axis

choose a test point on Real axis $s=0$,

$s=-2$. To the Right of this point the total no. of pole's and zero's is 1 which is an odd number

40. the Real axis b/w $s=0$ and $s=-2$ will be a part of Root locuss.

choose a test point on Real axis b/w $s=2$ & $s=-4$ to the right of this point the total no. of poles and zeros is 2 which is an even number. Hence the Real axis b/w $s=-2$ & $s=-4$ will not be a part of Root locuss.

choose a test point on Real axis b/w $s=-4$ & $s=-6$ to the right of this point the total no. of poles and zeros is 3 which is an odd number. Hence the Real axis b/w $s=-4$ and $s=-6$ will be a part of Root locuss.

Step - 3 To Find angle of Asymptotes of Centroid.

since there are 3 poles. The no. of root locus Branches are 3. The no. of Root locuss Branches are 3. There are no infinite zero. Hence all the 3 Root locuss Branches ends at zero at infinity.

$$\text{Angle of Asymptotes} = \frac{\pm 180(2q+1)}{n-m}$$

when $q=0$

$$\text{Angle of Asymptotes} = \frac{\pm 180[2(0)+1]}{3-0}$$

when $q_1=1$

$$\text{Angle of Asymptotes} = \frac{\pm 180(2 \times 1 + 1)}{3-0} = \pm 180^\circ$$

When $q=2$

$$\text{Angle of Asymptotes} = \pm \frac{180^\circ [2(2)+1]}{3-0} = \pm 60^\circ$$

when $q=3$

$$\text{Angle of Asymptotes} = \pm \frac{180^\circ [2(3)+1]}{3-0} = \pm 60^\circ = \pm 42^\circ$$

$$\begin{aligned}\text{Centroid} &= \frac{\text{sum of poles} - \text{sum of zeros}}{\text{Total no. of poles} - \text{total no. of zeros}} \\ &= \frac{0 + (-2) + (4) - 0}{3-0} = \frac{-2+4}{3} = \frac{2}{3}\end{aligned}$$

$$\boxed{\text{Centroid} = -2}$$

Step-4

To Find the Break away and Break in points

Assume Unity feedback system $H(s) = 1$

the closed loop transfer function

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{G(s)}{1+G(s)H(s)} \\ &= \frac{K/(s(s+2)(s+4))}{1+\frac{K}{s(s+2)(s+4)}}\end{aligned}$$

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{\frac{K}{s(s+2)(s+4)}}{s(s+2)(s+4) + K}\end{aligned}$$

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+2)(s+4) + K}$$

The characteristic equation is $s^3 + 6s^2 + 8s + K = 0$
unit-2, Pg-41/57

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$$K = -(s^3 + 6s^2 + 8s)$$

$$\frac{dK}{ds} = -[3s^2 + 12s + 8]$$

$$-(3s^2 + 12s + 8) = 0$$

$$3s^2 + 12s + 8 = 0$$

$$s_1 s_2 = \frac{-12 \pm \sqrt{144 - 4(3)(8)}}{2(3)}$$

$$= \frac{-12 \pm \sqrt{48}}{6}$$

$$s_1 s_2 = -0.845, -3.154$$

check for κ

$$\text{when } s = -0.845, \quad \kappa = -[(-0.845)^3 + 6(-0.845)^2 + 8(-0.845)]$$

$$\kappa = 3.08$$

since ' κ ' is positive and Real for $s = -0.845$

This is Break away point

check for κ

$$\text{when } s = -3.154, \quad \kappa = -[(-3.154)^3 + 6(-3.154)^2 + 8(-3.154)]$$

$$\kappa = -3.08$$

since ' κ ' is negative for $s = -3.154$

this is not a Break away point.

Step -5

To Find the angle of departure and arrival

Since there are no Complex poles and zeros we need not calculate the angle of departure and arrival

Step-6

To Find the crossing point on Imaginary axis

The characteristic equation is given by

$$s^3 + 6s^2 + 8s + K = 0$$

$$\text{putting } s = j\omega$$

$$(j\omega)^3 + 6(j\omega)^2 + 8(j\omega) + K = 0$$

$$-j\omega^3 + 6\omega^2 + 8j\omega + K = 0$$

$$-j\omega^3 + 8j\omega = -K$$

Imaginary point = 0

$$-6\omega^2 + K = 0$$

Real part = 0

$$-j\omega^3 + 8j\omega = 0$$

$$-6\omega^2 + K = 0$$

$$\omega^2 = 8$$

$$K = 48$$

$$\omega = \sqrt{8}$$

$$\omega = \pm 2\sqrt{2}$$

Step-7 To Find the value of K corresponding to

$$\frac{\omega}{Z} = 0.5$$

Given that

$$\frac{\omega}{Z} = 0.5$$

$$\text{let } \cos \phi = 0.5$$

$$\phi = \cos^{-1}(0.5)$$

$$[\phi = 60^\circ]$$

Draw a line op such that the angle b/w op & negative real axis is 60° as shown in unit-2, pg-43/57

4) graph the meeting point of the line up &

root locus.

$K = \frac{\text{product of length of vector from all poles}}{\text{product of length of vectors from all zeros}}$

$$K = \frac{l_1 l_2 l_3}{l_4 + l_5 + l_6 + l_7} \\ = \frac{1.3 \times 1.75 \times 3.5}{1.4 + 3.19 + 3.2 + 3.5} \\ K = 7.96 \approx 8$$

The openloop transfer function of a unity feedback system is given by $G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$ draw the Root locus of the system.

Step-1 :- To locate poles and zeros

the poles of the open loop T-F

$$s(s^2+4s+11) = 0$$

$$s_1 = 0 : s^2 + 4s + 11 = 0$$

$$s_1 = 0, s_2 s_3 = \frac{-4 \pm \sqrt{16 - 4(1)(11)}}{2(1)} \\ = \frac{-4 \pm \sqrt{16 + 44}}{2} \\ = \frac{-4 \pm \sqrt{12}}{2}$$

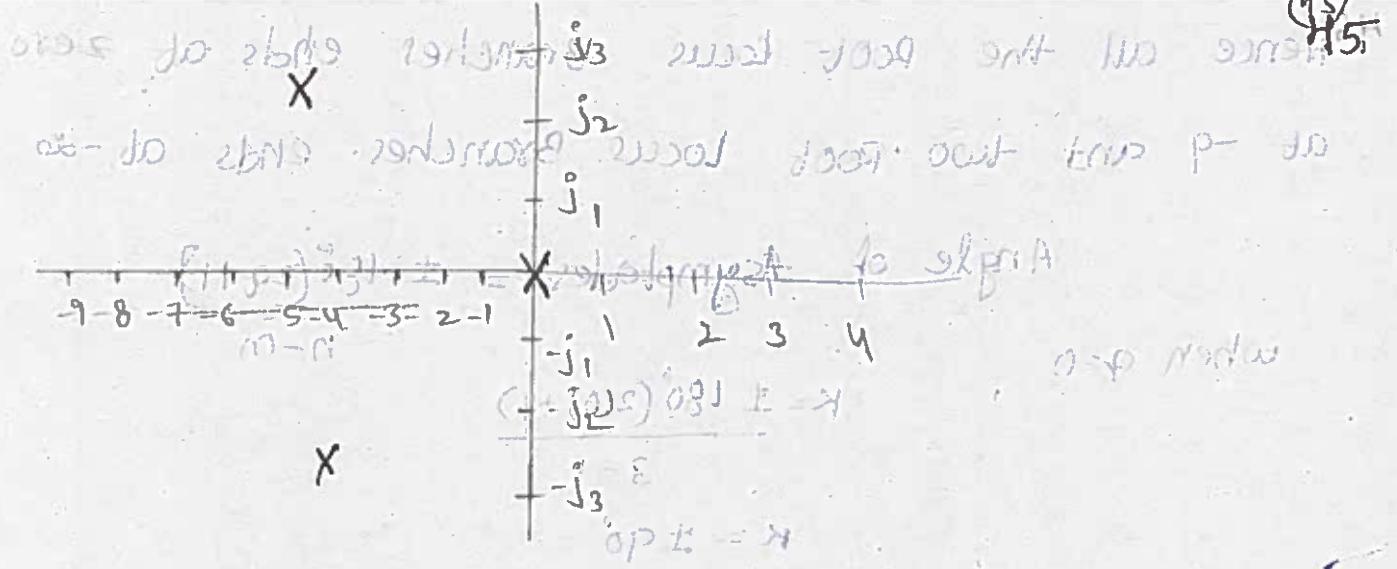
$$s_2 s_3 = -2 \pm j 2.64$$

$$s_1 = 0, s_2 = -2 + j 2.64, s_3 = -2 - j 2.64$$

Root Locus

and solve with help of a work

unit 2, Pg-44/52



- The zeros of the open loop transfer function is

$$s+q = 0$$

$$s = -q \quad \text{BPC} \pm = N$$

Step-2 To Find the Root's Locus on Real axis

There are one pole and one zero on Real axis choose a test point on Real axis b/w $s=0, s=-q$ to the Right of this point the total no.of poles and zero's is 1 which is on odd number, hence the Real axis b/w $s=0$ and $s=-q$ will be a part on Real axis

choose a test point on Real axis b/w $s=-q$ & $s=-\infty$ to the Right of this point the total no.of poles and zero's is 2. which is an even number. Hence the Real axis b/w $s=-q$ & $s=-\infty$ will not be a part on Real axis

Step-3

To Find the angle of Departure and centroid

since there are three poles. The no.of root locus Branches are 3. There is one finite zero

Hence all the Root locus Branches ends at zero at -9 and two Root Locus Branches ends at -20

$$\text{Angle of Asymptotes} = \pm \frac{180^\circ(2q+1)}{n-m}$$

when $q=0$

$$K = \pm \frac{180^\circ(2(0)+1)}{3-1}$$

$$K = \pm 90^\circ$$

when $q=1$

$$K = \pm \frac{180^\circ(2(1)+1)}{3-1} = P_1$$

$$K = \pm 270^\circ \quad P_2$$

when $q=2$

$$K = \pm \frac{180^\circ(2(2)+1)}{3-1}$$

$$K = \pm 450^\circ = (450-360)$$

$$K = \pm 90^\circ$$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{3-1}$$

$$= \frac{\text{total no. of poles} - \text{total no. of zeros}}{3-1}$$

$$= \frac{0 + (-2+j2.64) + (-2-j2.64) + 9}{3-1}$$

$$= \frac{0-2+j2.64-2-j2.64+9}{2}$$

$$= 5/2$$

$\boxed{\text{Centroid} = 2.5}$

Step - 4 $^\circ$

to find the Break-away and Break-in

points :-

Assume unity feedback system. $H(s) = 1$

$$\text{unit-2, } Ag = \frac{46}{57}$$

The closed loop transfer function

$$\begin{aligned}
 \frac{C(s)}{R(s)} &= \frac{G(s)}{1 + G(s)H(s)} \\
 &= \frac{\frac{K(s+9)}{s(s^2+4s+11)}}{1 + \frac{K(s+9)}{s(s^2+4s+11)}} \\
 &= \frac{(K \cdot \cancel{s}) K(s+9)}{s(s^2+4s+11) + K(s+9)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{C(s)}{R(s)} &= \frac{K(s+9)}{s^3 + 4s^2 + 11s + ks + 9k} \\
 &= \frac{K(s+9)}{s^3 + 4s^2 + (11+k)s + 9k}
 \end{aligned}$$

The characteristic equation is $s^3 + 4s^2 + (11+k)s + 9k = 0$

$$\begin{aligned}
 k &= -[s^3 + 4s^2 + (11+k)s] \\
 \frac{dk}{ds} &= -\frac{[s^3 + 4s^2 + (11+k)s]}{9}
 \end{aligned}$$

$$\frac{dk}{ds} = \frac{-[3s^2 + 8s + (11+k)]}{9}$$

$$-[3s^2 + 8s + (11+k)] = 0$$

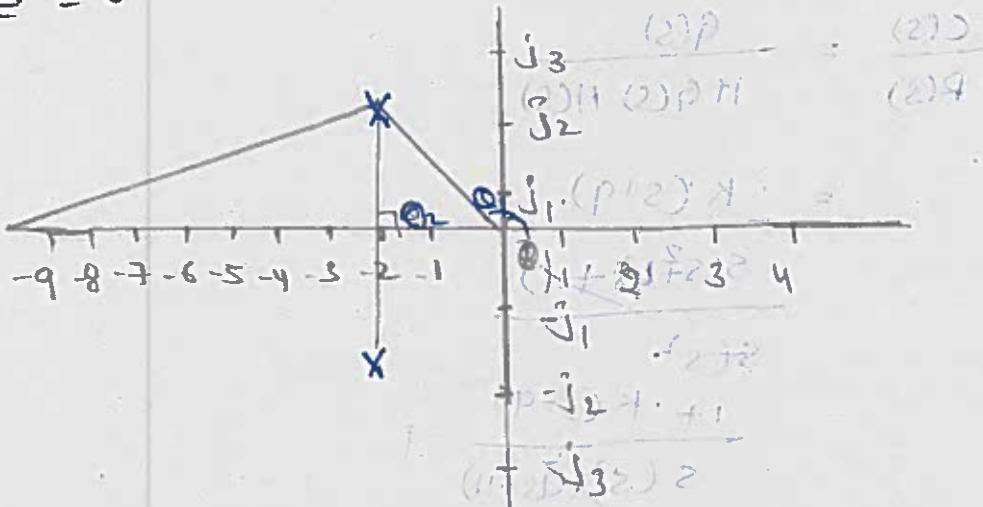
$$3s^2 + 8s + (11+k) = 0$$

from the location only one pole and one zero on real axis so no need to calculate break away and break in points

ai, bi branches \Rightarrow B ridges

$$0 = \lambda P + \omega i(\lambda + 11) + (\omega i)^2 \quad \text{Unit-2, Pg-47/57}$$

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Step 5 :-

$$\theta_1 = 180^\circ - \tan^{-1} \left(\frac{2.64}{2} \right)$$

$$\theta_1 = 127.14^\circ$$

$$\theta_2 = 90^\circ$$

$$\theta_3 = \tan^{-1} \left(\frac{2.64}{7} \right)$$

$$\theta_3 = 20.6^\circ$$

Angle of Departure ab pole A

$$= 180^\circ - (\theta_1 - \theta_2) + \theta_3$$

$$P = 180^\circ - (127.14^\circ + 90 + 20.6) + \theta_3$$

$$= 16.4^\circ$$

Angle of Departure ab complex plane

$$A.D.A^* = 16.4^\circ$$

Step 6

To find the crossing point on

Imaginary axis

The characteristic equation is given by

$$s^3 + us^2 + (11+k)s + 9k = 0$$

putting 's' is replaced by $j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + (11+k)j\omega + 9k = 0$$

$$j\omega^3 - (1+k)j\omega^2 - 4\omega^2 + (1+k)\omega + k = 0$$

$$\omega^2 + \frac{1}{k}\omega^2 = -(1+k)\omega$$

$$\omega^2 = -1 - k$$

$$\omega^2 = \pm 19.88$$

$$\omega = \pm 4.449$$

$$-4\omega^2 + 9k = 0$$

$$-4(1+k) + 9k = 0$$

$$-4 + 4k + 9k = 0$$

$$-4 + 5k = 0$$

$$-4 = -5k$$

$$k = \frac{4}{5}$$

$$k = 8.38$$

procedure for Constructing Root Locus

Step - 1: Location of poles and zeros

1. Draw the Real and Imaginary on an ordinary graph sheet and choose same scale both on Imaginary and Real axis.
2. the poles are marked by 'x' and zeros are marked by 'o' (circle)
3. The no. of Root's locus branches = no. of poles of open loop transfer function
4. The origin of a root locus is at a pole and end at zero
5. Let $n = \text{no. of poles}$, $m = \text{no. of zeros}$

Step - 2: Root locus on Real axis

1. In order to determine the part of Root locus on Real axis on Real axis, take a test point

on real axis if the total no. of pole's and zero's on the real axis to the right of this test point is odd number then the test point lies on the root locus.

If it is even then the test point does not lies on the root locus.

Step -3

Angle of Asymptotes and Centroid.

No. of Asymptotes is equal to no. of root locus Branch's gains to ~~poles and zeros~~ not ~~subtended~~

The angles of Asymptotes and centroid are given by the following formula

$$\text{Angle of Asymptotes} = \frac{\pm 180(2q+1)}{n-m}$$

where $q = 0, 1, 2, 3, \dots$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{\text{Total no. of poles} - \text{total no. of zeros}}$$

Step -4 Break away and Break in points

The Break away or Break in point's either lie on real axis (or) complex conjugate planes

If there is a root locus on real axis b/w two poles then there exist break away point

If there is a root locus on real axis b/w two zeros then there exists a break in point

If there is a root locus on real axis only blw pole and zero than there may be (or) may not be Breakaway (or) Break-in points write the characteristic eqn. be in the form $B(s) + kA(s) = 0$

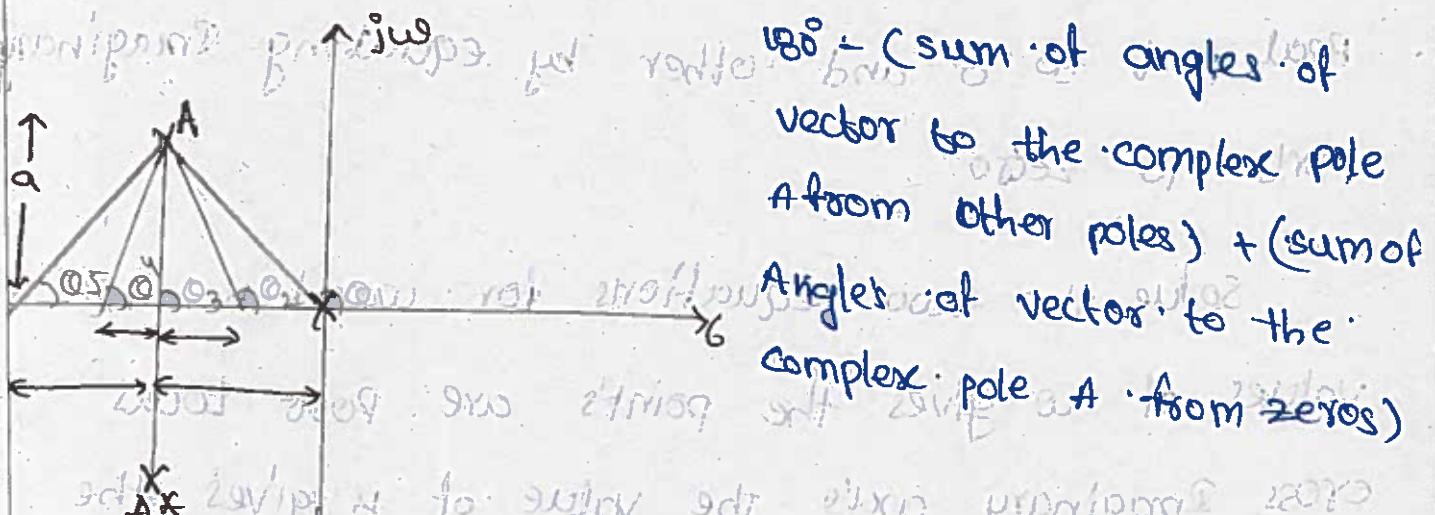
$$K = \frac{B(s)}{A(s)}$$

The Break away and Break-in point is given by the Root's in to point providing for this value of Root

The gain 'k' should be positive and Real

Step = 50 Angle of Departure and Angle of arrival

Angle of Departure from Complex pole A =



$$\Theta_1 = 180^\circ - \tan^{-1}\left(\frac{a}{b}\right)$$

$$\Theta_2 = 180^\circ - \tan^{-1}(a/c)$$

$$\Theta_3 = 90^\circ$$

$$\Theta_4 = \tan^{-1}(a/y)$$

$$\Theta_5 = \tan^{-1}(a/c)$$

$$\text{Angle of Departure from pole } A = 180^\circ - (\Theta_1 + \Theta_3 + \Theta_5)$$

$$\text{Angle of Arrival} = 180^\circ - (\Theta_2 + \Theta_4)$$

Angle of arrival at Complex zero.

$A = 180^\circ - (\text{sum of angles of vectors to the complex pole zero} + \text{from other zeros}) + (\text{sum of angles of vectors to the complex '0' from poles})$

Step - 6 point of intersection of Root Locus with Imaginary axis

from the characteristic eqn letting $s = i\omega$ and separate real part and imaginary part

Two equations are obtained by equating real part to '0' and other by equating imaginary part to zero

Solve the two equations for ω and ' k ', the values of ω gives the points where Root locus cross Imaginary axis the value of ' k ' gives the value of gain ' k ' at their crossing point also this value of ' k ' is the stability of the system

Step - 7 test points and Root loci

choose the point using a protractor estimate the angle's of vector drawn to this point and adjust the point to satisfy angle criterion

Repeat the procedure for few more test points
 Draw the Root loci's form the knowledge of graph
 and the system information obtained in step 1
 through 5

* Routh Hurwitz Criterion :-

The Routh Hurwitz stability criterion is an analytical procedure to determine whether all the roots of a polynomial have negative real parts or not.

When all the coefficients are positive, the system is not necessarily stable. Even though the coefficients are positive sum of the roots may lie on the right of s-plane (or) on the imaginary axis.

In order for all the roots to have negative real parts it is necessary but not sufficient that all coefficients of the characteristic equation be positive.

Construct of Routh array :-

Let a characteristic polynomial be

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n$$

The coefficients of the polynomial are arranged

unit in two rows was shown below

Step 10 Substituting value of $s = 0$ in eq. of unit

$$S^n : a_0 \ a_2 \ a_4 \ a_6 \ \dots$$

in next row becomes non dominant make unit below

$$S^{n-1} : a_1 \ a_3 \ a_5 \ \dots$$

$$S^{n-2} : x_0 \ x_1 \ x_2 \ \dots$$

$$S^{n-3} : y_0 \ y_1 \ y_2 \ \dots$$

$$S^{n-4} : \text{symbols of sequence - positions in}$$

$$\text{order - last symbol is to zeros - unit}$$

$$\dots$$

$$\text{unit } S^0 \text{ has one zero - } 1 - \text{ unit}$$

In the next row, find $\frac{a_0 + a_2}{2}$ for 2nd row

In the construct of Routh array one may

come across the following three cases.

Case - 1

Normal Routh array

Case - 2 for first problem in JEE Main 2018

A row of all zeros.

Case - 3

first element of a row is 0 but

other elements are not zero.

Case - 1

Normal Routh Array :-

In this case there is no difficulty in forming Routh array. The Routh array can be constructed as explained above. The sign changes are not noted to find the no. of roots lying on right half of s-plane and the stability of the system can be estimated.

If there is no sign change's in the first column in the first column in the Routh array than all the roots are lying on the left half of s-plane and the system is stable.

If there is sign change's in the first column of the Routh array then the system is unstable and the no. of root's are lying on the right half of s-plane is equal to the no. of sign change's. The remaining root's are lying on left half of s-plane.

Case - 2

A Row of All zeros

1. Determine the auxiliary polynomial, $A(s)$
2. Differentiate the auxiliary polynomial w.r.t s
3. The row of zero's is replaced with

$$\frac{d}{ds} [A(s)]$$

4. Continue the construction of Routh array in the useful manner that is case -1
5. similarly as case -1 digit no page 21001

case -3

First element of a row is zero but other elements are not zero.

By constructing Routh array if a zero is encountered as first element of a row then all the elements of the next row will infinite.

To overcome this problem let $0 \rightarrow \epsilon$ and complete the construction of Routh array in the useful manner that is case -1.

finally let $\epsilon \rightarrow 0$ and determine the values of elements of the Routh array which are functions of ϵ .

As follows as case -1

Frequency domain specification :-

The performance and characteristic of a system in frequency domain are measured in terms of frequency domain specifications.

The frequency domain specifications are

1. Resonant peak, m_x
2. Resonant frequency, ω_x
3. Band width w_b
4. cut off rate
5. Gain margin, k_g
6. Phase margin, ϕ

=====

Leighfieldia gigantea speciosa

of the differentiation and development
of the bulbous rhizome and bulbils in some of
the varieties of the genus Leighfieldia are described
as follows:

Leighfieldia bifolia:

Rhizome lengthened and swollen,

bulbils white yellowish.

Cup all green.

petioles black.

Leaves white.

* Frequency Response Analysis *

The frequency response is the steady state response of a system when the i/p of the system is a Sinesoidal signal.

Consider a Linear time invariant system is shown in figure.



$$\text{Let } x(t) = X \sin \omega t$$

Then o/p $y(t)$ be also a Sinesoidal Signal of same frequency but with different magnitudes and phase angles

$$y(t) = Y \sin(\omega t + \phi)$$

In a linear time invariant system the frequency response is independent of the amplitude and phase of the i/p signal.

The frequency response of a system is normally obtained by varying the frequency of the i/p signal by keeping the magnitude of i/p signal at a constant value.

In the system $t=f(T(s))$, if 's' is replaced by $j\omega$ then the resulting $t-f$ is $T(j\omega)$ is called Sinesoidal t.f.

$$\text{Per loop t.f } G(j\omega) = |G(j\omega)| L(j\omega)$$

$$\text{loop t.f } G(j\omega) H(j\omega) = |G(j\omega)H(j\omega)| / |G(j\omega) + H(j\omega)|$$

$$\text{closed loop t.f } \frac{C(j\omega)}{R(j\omega)} = |M(j\omega)| L(j\omega)$$

$$\text{where } M(j\omega) = \frac{G(j\omega)}{(1 + G(j\omega)H(j\omega))}$$

Frequency domain specification:-

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specification

They are explained in below

Resonant peak (M_r):-

The maximum value the magnitude at closed.

loop t.f is called Resonant peak (M_r) A large resonant peak corresponds large over shoot in transient response

Resonant frequency :-

The frequency at which the resonant peak occurs is called Resonant frequency (ω_r). This is related to the frequency of oscillation in step response. The resonant frequency is indicated by transient response.

Band width (B.W) :-

The B.W is the range of frequencies for which the system gain is more than -3dB. The frequency at which the gain is -3dB is called Cut-off frequency.

The band width is usually defined for closed loop system and it transmits the signal whose frequencies are less than the Cut-off frequency. The B.W is measured by the ability of feed back system to reproduce the I/P signal.

Cut-off rate :-

The slope of log magnitude curve near the Cut-off frequency is called the Cut-off rate. The Cut-off rate indicates the ability of the system to difference b/w the signal from noise.

Gain Margin (Kg) :-

The gain margin in Kg is defined as the reciprocal of magnitudes of open loop t.f at phase cross frequency.

The frequency at which the phase of open loop t.f is -180° is called phase cross over frequency ω_{pc} .

$$\text{therefore gain margin (Kg)} = \left| \frac{1}{G(j\omega_{pc})} \right|$$

The gain margin in db can be expressed as Kg in db.

$$= 20 \log K_g = 20 \log \left| \frac{1}{G(j\omega_{pc})} \right|$$

$$= -20 \log |G(j\omega_{pc})|$$

Phase Margin :-

The phase margin is that amount of additional phase lag at a gain cross over frequency required to bring the system to range of instability.

The gain cross frequency ω_{c} is the frequency at which the magnitude at the open loop $|T|$ is unity.

The phase margin γ is obtained by adding 180° to the phase angle ϕ of the open loop $|T|$ at a gain cross over frequency.

Therefore phase gain $\gamma = 180^\circ + \phi_{g_c}$

Derivation of frequency domain specifications! - (for 2nd order system)

1) Response Resonant peak (M_r) :-

Consider the closed loop $|T|$ of 2nd order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\text{Put } s = (j\omega)$$

$$\begin{aligned} T(j\omega) &= \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{-\omega^2 + 2j\zeta\omega_n\omega + \omega^2} \\ &= \frac{\omega_n^2}{\omega_n^2 \left(-\frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n} + 1 \right)} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}} \\ &= \frac{1}{1 - 4^2 + j2\zeta 4} = \frac{1}{\sqrt{(1-4^2) + 4\zeta^2 4^2}} \quad \frac{1}{\tan^{-1}\left(\frac{2\zeta 4}{1-4^2}\right)} \end{aligned}$$

$$M_r = \left[(1-4^2) + 4\zeta^2 4^2 \right]^{-1/2}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta 4}{1-4^2}\right) \quad \text{where } u = \frac{\omega}{\omega_n}$$

$$\frac{dm}{du} = 0$$

$$\begin{aligned} \frac{dm}{du} &= -\frac{1}{2} \left[(1-4^2) + 4\zeta^2 4^2 \right] - \frac{3}{2} \left[2(1-4^2)(-24) + 8\zeta^2 4^2 \right] = 0 \\ &= 44(1-4^2) - 8\zeta^2 4^2 = 0 \end{aligned}$$

$$4^2 = 1 - 2\zeta^2$$

$$4 = \sqrt{1 - 2\zeta^2}$$

$$M_r = \left[(1-4^2) + 4\zeta^2 4^2 \right]^{-1/2} / 4 = 4^2 \quad \text{unit-3, Pg-3/41}$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\xi^2}}$$

Resonant frequency (ω_r):-

$$\text{Let } \omega = \frac{\omega}{\omega_n}$$

$$\zeta_r = \frac{\omega_r}{\omega_n}$$

$$\omega_r = \omega_n \zeta_r$$

$$\omega_r = \omega_n \sqrt{1-2\xi^2_r}$$

Band width (ω_b):-

Let normalised band width equal to $ub = \frac{\omega_b}{\omega_n}$

$$\omega_b = ub \times \omega_n$$

$$\zeta_l = \zeta R b$$

Magnitude of the closed loop system is $\frac{1}{\sqrt{2}}$ or -3db hence
in eqn for M putting $\omega = ub$ and equating to $\frac{1}{\sqrt{2}}$

$$M = \frac{1}{\sqrt{2}} = \frac{1}{[(1-4^2) + 4\xi^2 4^2]^{1/2}}$$

$$[1-4^2] + 4\xi^2 4^2 = 2 \Rightarrow 4b^4 - 24b^2(1-2\xi^2) - 1 = 0$$

$$4b^2 = x$$

$$x^2 - 2x(1-2\xi^2) - 1 = 0$$

$$a=1 \quad b=-2(1-2\xi^2) \quad c=-1$$

$$ub = \sqrt{x}$$

$$ub = \sqrt{1-2\xi^2} + 4\xi^2(\xi^2 - 1) + 2$$

$$\omega_b = \omega_n ub$$

$$\omega_b = \omega_n [\sqrt{1-2\xi^2} + 4\xi^2(\xi^2 - 1) + 2]$$

$$\omega_b = \omega_n [1-2\xi^2 + \sqrt{2-4\xi^2 + 4\xi^4}]^{1/2}$$

⑤

Phase Margin (V) :-

The open loop t.f of 2nd order system is

$$T(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s}$$

$$s = j\omega$$

$$T(j\omega) = \frac{\omega_n^2}{j\omega(j\omega + 2\xi\omega_n)} = \frac{\omega_n^2}{\omega_n^2 \left[-\frac{\omega}{\omega_n} + \frac{2j\xi\omega}{\omega_n} \right]}$$

$$= \frac{1}{-\left[\frac{\omega}{\omega_n}\right]^2 + j2\xi\frac{\omega}{\omega_n}} \quad \text{let } u = \frac{\omega}{\omega_n}$$

$$T(j\omega) = \frac{1}{-4^2 + 2j\xi_4}$$

$$\text{Magnitude} = \frac{1}{\sqrt{4^2 + 4\xi_4^2}} \tan^{-1} \left(\frac{2\xi_4}{-4} \right)$$

$$M = [4^2 + 4\xi_4^2]^{1/2}$$

$$\phi = -\tan^{-1} \left[\frac{2\xi_4}{-4} \right] = \tan^{-1} \left(\frac{2\xi_4}{4} \right)$$

gain cross over frequency of $G(j\omega) = 1$.

$$u_{gc} = \frac{\omega_{gc}}{\omega_n}$$

$$u = u_{gc}$$

$$G(j\omega) = \left[4g_c^4 + 4\xi^2 u^2 g_c \right]^{1/2}$$

$$1 = \frac{1}{\sqrt{4g_c^4 + 4\xi^2 u^2 g_c}}$$

$$4g_c^4 + 4\xi^2 u^2 g_c = 1$$

$$u_{gc} = 1$$

$$u g_c = \sqrt{1}$$

$$\xi^2 + 4\xi^2 - 1 = 0$$

$$\xi = -2\xi \pm \sqrt{4\xi^2 + 1}$$

$$u g_c = \sqrt{-2\xi \pm \sqrt{4\xi^2 + 1}}$$

$$V = 180 + LG(j\omega) / \omega = w g_c \quad u = u g_c$$

$$LG(j\omega)$$

$$V = 180^\circ + \tan^{-1} \left[\frac{2\xi}{\sqrt{-2\xi^2 + \sqrt{4\xi^2 + 1}}} \right]$$

Frequency Response Plot :-

frequency response analysis of C.S can be carried either analytically or graphically various graphical techniques are

1. Bode plot 2. Polar plot 3. Nichols plot

4. Mand N Circle 5. Nichols chart are usually drawn for open loop system. for the open loop response plot the performance and stability of closed loop systems are estimated.

The Mand N Circle and Nichols chart are used to graphically determine the frequency response of unity feed back closed loop system from the knowledge of open loop response.

The frequency response plots are used to determine the frequency domain specification to study the stability of systems and to exist the gain of the system to satisfy the desired specifications.

The Bode plot is a frequency response plot of all of a system

The bode plot consisting the two graphs.

1. Magnitude graph (drawn b/w magnitude and log w).
2. Phase graph (drawn b/w phase angle and log w).

The bode plot can be drawn for both open & closed loop i.e. if usually $G(j\omega)$ is $20 \log |G(j\omega)|$ were the base of log is 10 then unit of magnitude is db

The curves are drawn on semi log graph using long scale the magnitude graph is straight line. the phase graph is near curve the main advantage is multiplication of magnitude can be converted into addition

⑦ Consider the open loop $t \cdot t G(s) = \frac{K(1+sT_1)}{(1+sT_2)(1+sT_3)s}$

$$s = j\omega$$

$$\begin{aligned} G(j\omega) &= \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)(1+j\omega T_3)} \\ &= \frac{K \sqrt{1+\omega^2 T_1^2} \tan^{-1}(\omega T_1)}{\omega L 90 \sqrt{1+\omega^2 T_2^2} \tan^{-1}(\omega T_2) \sqrt{1+\omega^2 T_3^2} \tan^{-1}(\omega T_3)} \\ A = |G(j\omega)| &= \frac{K \sqrt{1+\omega^2 T_1^2}}{\omega \sqrt{1+\omega^2 T_2^2} \sqrt{1+\omega^2 T_3^2}} \end{aligned}$$

$$\text{Phase angles} = \tan^{-1}(\omega T_1) - 90^\circ \cdot \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$$

$$A = |G(j\omega)| = 20 \log |G(j\omega)| = 20 \log \frac{K \sqrt{1+\omega^2 T_1^2}}{\omega \sqrt{1+\omega^2 T_2^2} \sqrt{1+\omega^2 T_3^2}}$$

$$|A| = 20 \left[\log K \sqrt{1+\omega^2 T_1^2} - \log \omega \sqrt{1+\omega^2 T_2^2} - \sqrt{1+\omega^2 T_3^2} \right]$$

Basic factor of $G(j\omega)$:-

① Constant gain K:-

$$G(s) = K$$

$$s = j\omega$$

$$G(j\omega) = K 10^\circ$$

$$A = |G(j\omega)| = K$$

$$\phi = 0^\circ$$

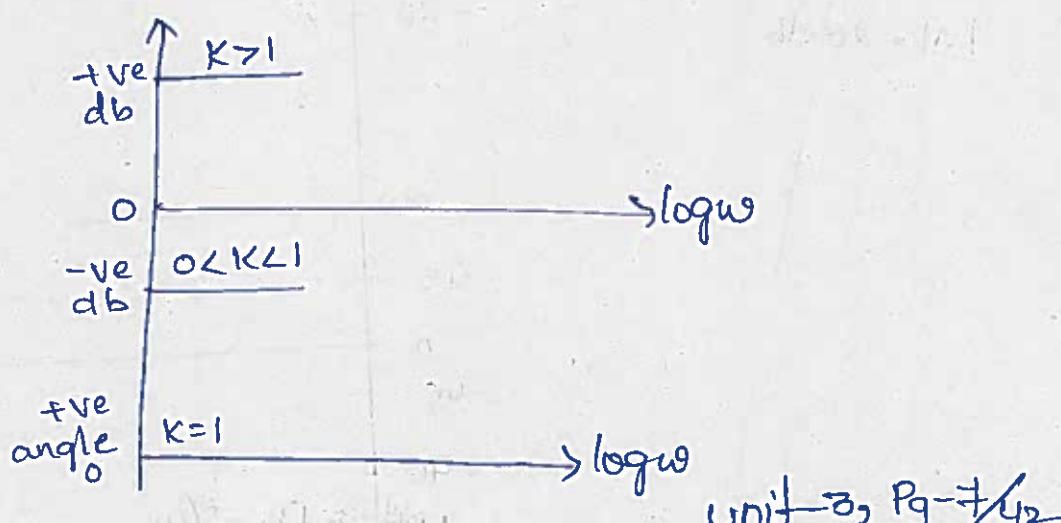
$$K=0 = 20 \log 0 = 1$$

$$K=1 = 20 \log 1 = 0$$

$$K>1 = 20 \log (K) = +db$$

$$\text{When } 0 < K < 1, \quad K = -db$$

Graph :-



(ii) Integral factor ($\frac{K}{j\omega}$) :-

$$G(s) = \frac{K}{s} ; s = j\omega$$

$$|G(j\omega)| = \frac{K}{j\omega}$$

$$\text{magnitude} = \frac{K}{\omega}$$

$$\text{phase} = -90^\circ$$

$$\text{magnitude in db is } 20 \log \frac{K}{\omega} = 20 [\log K - \log \omega]$$

$$\omega = 0.1K$$

$$-A = 20 \text{ db}$$

$$\omega = K$$

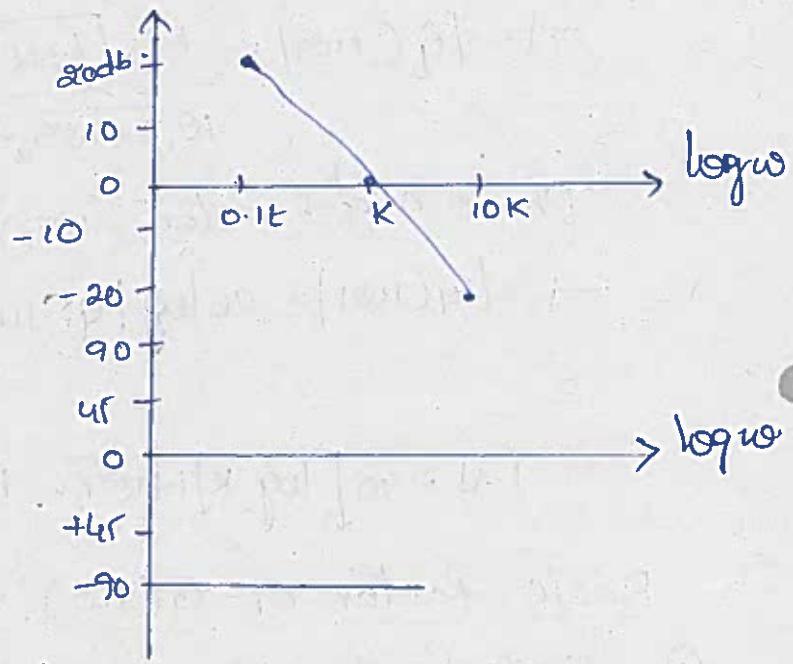
$$-A = 0 \text{ db}$$

$$\omega = K$$

$$-A = 0 \text{ db}$$

$$\omega = K$$

$$-A = -20 \text{ db.}$$



(iii) Derivative factor ($Kj\omega$) :-

$$G(s) = KS ; s = j\omega$$

$$-A = |G(j\omega)| = K\omega \angle 90^\circ$$

$$\text{magnitude} = K\omega$$

$$\text{Phase} = 90^\circ$$

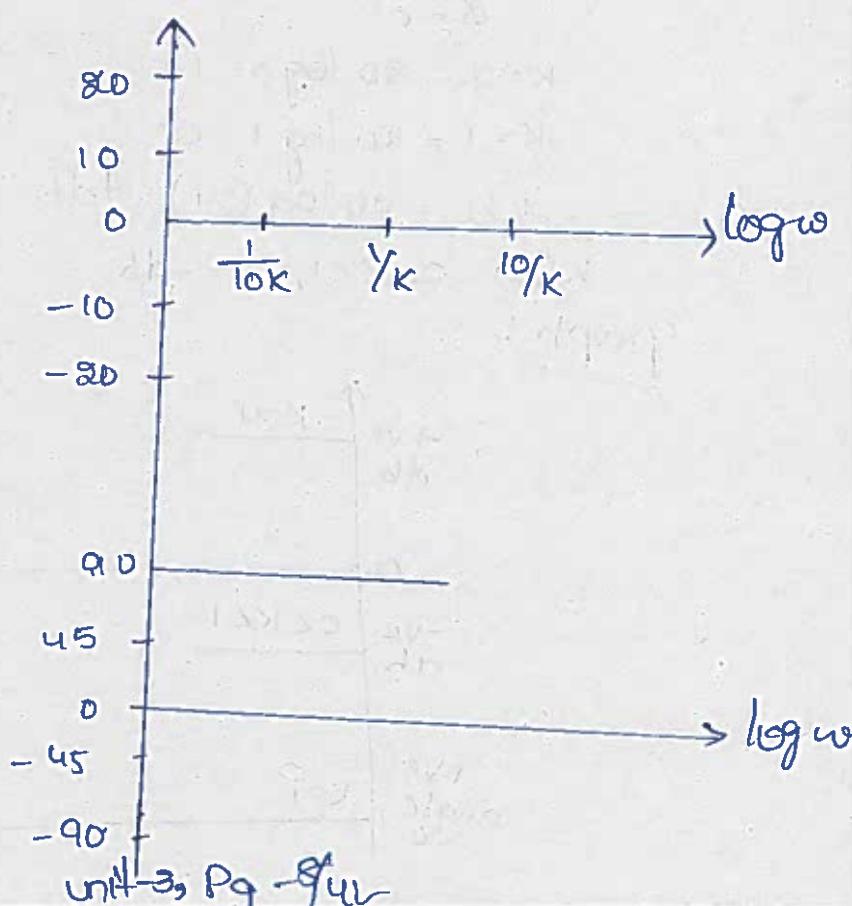
$$\text{at } \omega = \frac{1}{10K}$$

$$-A = -20 \text{ db}$$

$$\text{at } \omega = Y_K = 0 \text{ db}$$

$$\text{at } \omega = 10/K$$

$$|A| = 20 \text{ db}$$



Produce -for magnitude graph:-

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Step 1:- Convert the $t \cdot f$ into bode form or time constant form

$$G(s) = \frac{k(1+sT_1)}{s(1+sT_2)(1+sT_3)} \quad \text{put } s=j\omega$$

Step 2:- List the corner frequency in the increasing order and prepare a table as follows.

Term	Corner frequency	slope	change in slope.

enter k (or) $(k/j\omega)$ or $(kj\omega)$ as first terms and the other terms the increasing order of corner frequencies then enter the corner frequency the slope of each term and change in slope at every corner frequency.

Step 3:- Choose an arbitrary frequency ω_1 which is lesser than the lowest corner frequency. Calculate the db magnitude of k or $(k/j\omega)$ or $(kj\omega)$ at ω_1 and at the ω_1 lowest corner frequency.

Step 4:- Then calculate the gain at every corner frequency at one by using the formula.

$$\text{gain at } \omega_2 = \text{change in gain } (\omega_1 \text{ to } \omega_2) + \text{gain at } \omega_1.$$

$$\omega_2 = \text{slope from } \omega_1 \text{ to } \omega_2 \times \frac{\log \omega_2}{\log \omega_1} + \text{gain at } \omega_1.$$

Step 5:-

Choose an arbitrary frequency ω_H which is greater than the highest corner frequency. Calculate the gain at ω_H by using the formula in step 4.

Step 6:-

In a semi log graph the required range of frequency on x-axis and range of db magnitude on y-axis after choosing proper scale

Step 7:-

Mark all the points obtained the steps 3, 4, 5 on graph and join the points by straight lines, mark the slope at every part of graph.

1. For the following t.f draw the bode plot and determine the system gain cross over frequency to be 5 rad/sec.

$$G(s) = \frac{ks^2}{(1+0.2s)(1+0.02s)}$$

Sol :- Put $s = j\omega$

$$G(j\omega) = \frac{k(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

$$\omega_{c1} = \frac{1}{T_1} = \frac{1}{0.2} = 5 \text{ rad/sec}; \quad \omega_{c2} = \frac{1}{T_2} = \frac{1}{0.02} = 50 \text{ rad/sec}$$

Table :-

Term	Corner frequency	slope	change in slope.
$(j\omega)^2$	-	40 db	-
$\frac{1}{1+0.2j\omega}$	$\omega_{c1} = 5 \text{ rad/sec}$	-20 db	$40 - 20 = 20 \text{ db}$
$\frac{1}{1+0.02j\omega}$	$\omega_{c2} = 50 \text{ rad/sec}$	-20 db	$20 - 20 = 0 \text{ db}$

low Corner frequency $\omega_c < \omega_{c1}$.

high Corner frequency $\omega_H > \omega_{c2}$

$$\omega_1 = 0.5 \text{ rad/sec}$$

$$\omega_H = 100 \text{ rad/sec}$$

at $\omega = \omega_L$

$$-A = 20 \log |G(j\omega)|$$

$$= 20 \log \omega'$$

$$= 20 \log (0.5)^2$$

$$= -12 \text{ db}$$

at $\omega = \omega_{c1}$

$$-A = 20 \log (5^2)$$

$$-A = 20 \text{ db}$$

" Calculation of K :-

Given that the Gain Cross Over frequency is 5 rad/sec
at $\omega = 5 \text{ rad/sec}$ the gain $A = 28 \text{ db}$

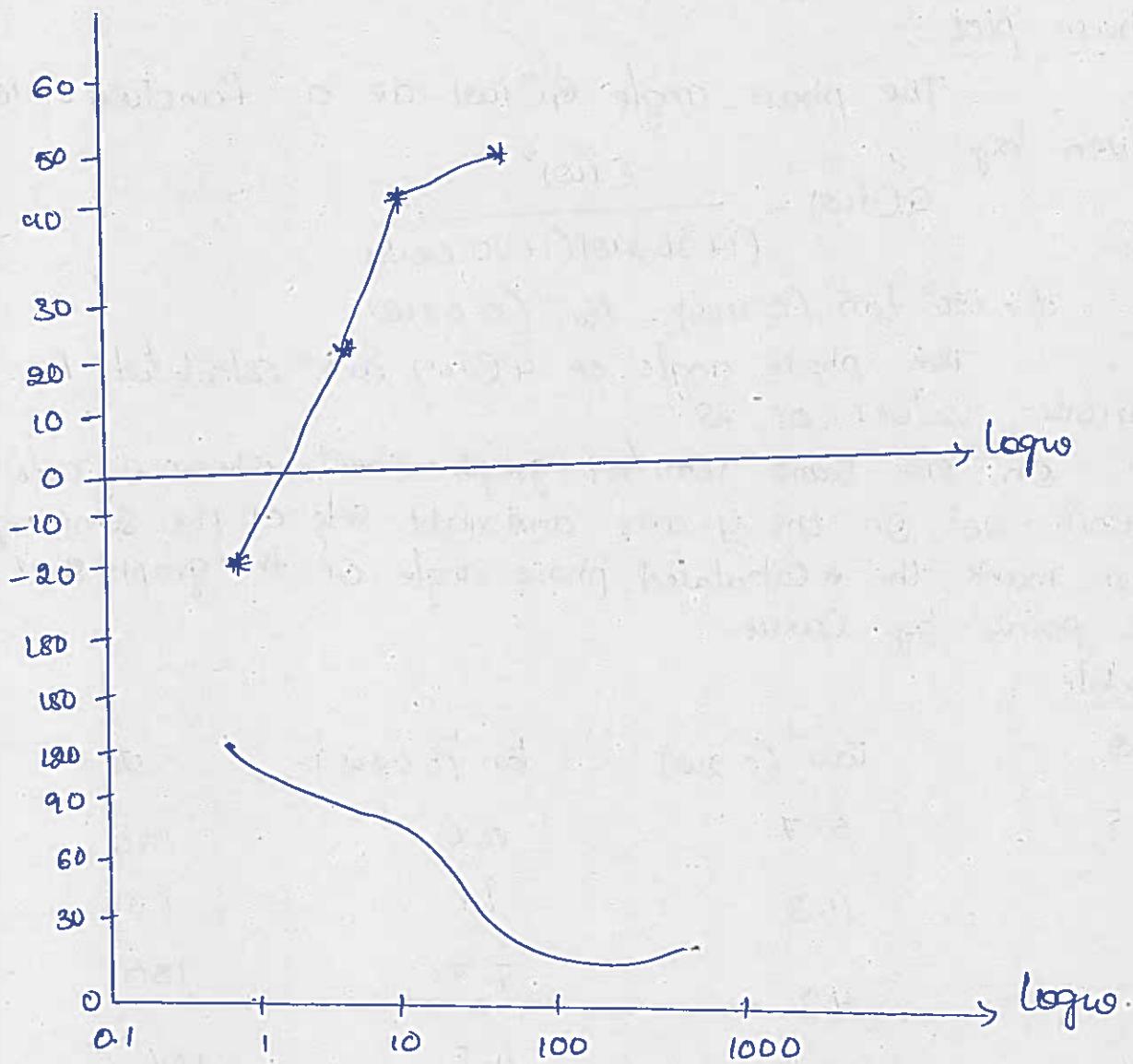
In gain Cross over frequency is 5 rad/sec then at that frequency that the db gain should be zero, i.e. -28 db hence to every point of magnitude plot a db gain of -28 db should be added

The value of K is calculated by equating to $20 \log K$ to -28 db therefore $20 \log K = -28$.

$$\log K = \frac{-28}{20} = -1.4$$

$$K = 0.0398$$

Graph:-



Let $\omega = \omega_{C_2}$

$$\begin{aligned} A &= \left[\omega_{C_1} \text{ to } \omega_{C_2} \cdot \log \frac{\omega_{C_2}}{\omega_{C_1}} \right] + A_{\omega_{C_1}} \\ &= \left[20 \log \left[\frac{50}{5} \right] + 28 \right] \end{aligned}$$

$$A = 48 \text{ dB}$$

At $\omega = \omega_H$

$$A = \left[\omega \log \left[\frac{100}{50} \right] \right] + 48.$$

$$A = 48 \text{ dB}$$

Let the points a, b, c and d be the points corresponding to frequencies $\omega_L, \omega_{C_1}, \omega_{C_2}$ and ω_H respectively on the magnitude plot in a semi log graph sheet choose a scale of 1 unit = 10dB on Y-axis the frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scale on X-axis. Fix the points a, b, c and d on the graph join the points by straight lines.

Phase plot :-

The phase angle $\phi(j\omega)$ as a function of ω is given by

$$G(j\omega) = \frac{(j\omega)^v}{(1+j0.2\omega)(1+j0.02\omega)}$$

$$\phi = 180^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega)$$

The phase angle of $G(j\omega)$ are calculated for various values of ω .

On the same semi log graph sheet choose a scale of 1 unit = 20° on the Y-axis and right side of the semi log sheet mark the calculated phase angle on the graph sheet join the points by curve.

Table :-

ω	$\tan^{-1}(0.2\omega)$	$\tan^{-1}(0.02\omega)$	ϕ
0.5	5.7	0.6	174
1	11.3	1.1	168
5	45	5.7	180
10	63.4	11.3	106
50	84.3	45	50
100	87.1	63.4	430

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1. Resonant Peak, mr.

The Maximum Value of the Magnitude of a closed loop transfer function is called the resonant peak, mr or the large resonance peak corresponding to a large overshoot in a transient response.

2. Resonant frequency :-

The frequency at which the resonant peak occurs is called Resonant frequency ω_r . This is related to the frequency of oscillations in the step response.

3. Band width ω_b :-

The Band width is the range of frequencies for which normalized gain of the system is more than -3dB. The frequency at which the gain is -3dB is called Cut-off frequency.

4. Cut off Rate :-

The slope of the log magnitude curve near the cut-off frequency is called Cut-off Rate. The cut-off rate indicates the ability of the system to distinguish the system from noise.

5. Gain Margin K_g :-

The Gain Margin K_g is defined as the value of gain to be added to system in order to bring the system to the edge of instability

$$\text{the Gain Margin } K_g = \frac{1}{|G(j\omega_{pc})|}$$

The Gain Margin in dB can be expressed as

$$K_g \text{ in dB} = 20 \log K_g$$

$$= 20 \log \left| \frac{1}{G(j\omega_{pc})} \right|$$

6. Phase Margin ϕ :-

The phase margin ϕ is defined as the additional phase lag to be added to the gain cross over frequency in order to bring the system to the edge of instability. The gain cross over frequency ω_{co} is the frequency at which open loop transfer fun. is unity.

∴ The phase margin γ is obtained by adding 180° to the phase angle ϕ of the open loop transfer fun at the gain cross over frequency

$$\therefore \text{phase margin } \gamma = 180^\circ + \phi_{gc}$$

~~Frequency Domain Specification~~

Frequency Domain specification of second order system :-

1. Resonant Peak (M_r) :-

Consider the closed loop transfer fun of second order system is given by.

$$M(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The sinusoidal transfer fun $M(j\omega)$ is obtained by letting $s = j\omega$.

$$M(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$$= \left[\frac{\omega_n^2}{\omega_n^2 \left[-\left(\frac{\omega}{\omega_n}\right)^2 + j2\xi\frac{\omega}{\omega_n} + 1 \right]} \right] \quad \left[\because u = \frac{\omega}{\omega_n} \right]$$

$$M(j\omega) = \frac{1}{1 - u^2 + j2\xi u}$$

where

u = Normalised frequency

$$M(j\omega) = \frac{1}{(1-u^2) + j2\xi u}$$

Let

M = Magnitude of closed loop transfer fun.

α = Phase of closed loop transfer fun

$$M = |M(j\omega)| = \sqrt{(1-u^2)^2 + 4\xi^2 u^2}$$

$$M = \sqrt{(1-u^2)^2 + 4\xi^2 u^2}^{1/2}$$

$$\alpha = \tan^{-1} \left(\frac{2\xi u}{1-u^2} \right)$$

$$\alpha = \tan^{-1} \left(\frac{2\xi u}{1-u^2} \right)$$

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The Resonant peak is the max value 'm' & the condition for max value of 'm' can be obtained by differentiating the eqn of m w.r.t 'u' and letting

$$\frac{dm}{du} = 0 \quad \text{when } u=u_r$$

where

$$u_r = \frac{\omega_r}{\omega_n} = \text{Normalised Resonant frequency}$$

$$M = [(1-u)^2 + 4\xi^2 u^2]^{1/2}$$

Diff w.r.t u

$$\frac{dm}{du} = -\frac{1}{2} [(1-u)^2 + 4\xi^2 u^2]^{-3/2} [2(1-u)(-2u) - 8\xi^2 u]$$

$$= \frac{8u(1-u) + 4\xi^2 u}{[(1-u)^2 + 4\xi^2 u^2]^{3/2}}$$

$$0 = 8u(1-u) + 4\xi^2 u$$

Replace 'u' by u_r in the above eqn then.

$$\frac{8u_r(1-u_r) + 4\xi^2 u_r}{[2(1-u_r)^2 + 4\xi^2 u_r^2]^{3/2}} = 0$$

$$8u_r(1-u_r) + 4\xi^2 u_r = 0$$

$$8u_r - 8u_r^2 - 4\xi^2 u_r = 0$$

$$4u_r^2 = 8u_r - 4\xi^2 u_r$$

$$u_r^2 = 1 - 2\xi^2$$

$$u_r = \sqrt{1 - 2\xi^2}$$

At Resonant peak Condition $M=M_r$ and $u=u_r$.

$$M = \frac{1}{[(1-u)^2 + 4\xi^2 u^2]^{1/2}}$$

$$M_r = \frac{1}{[(1-u_r)^2 + 4\xi^2 u_r^2]^{1/2}}$$

$$M_r = \frac{1}{[(1-(1-2\xi^2))^2 + 4\xi^2(1-2\xi^2)]^{1/2}}$$

$$M_r = \frac{1}{[4\zeta^4 + 4\zeta^2 - 8\zeta^2]^{\frac{1}{2}}}$$

$$M_r = \frac{1}{[4\zeta^2 - 4\zeta^2]^{\frac{1}{2}}}$$

$$M_r = \frac{1}{[4\zeta^2(1 - 3\zeta^2)]^{\frac{1}{2}}}$$

$$M_r = \frac{1}{2\zeta\sqrt{1 - 3\zeta^2}}$$

$$\therefore \text{Resonant peak, } M_r = \frac{1}{2\zeta\sqrt{1 - 3\zeta^2}}$$

(ii) Resonant frequency (ω_r) :-

$$\text{Normalised Resonant frequency } u_r = \frac{\omega_r}{\omega_n}$$

$$\omega_r = \omega_n u_r$$

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

$$\text{Resonant frequency } \omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

Band width (wb) :-

$$\text{Let normalised Band width } u_b = \frac{\omega_b}{\omega_n}$$

when, $u = u_b$ the magnitude "M" of the closed loop system is $\frac{1}{\sqrt{2}}$ (or) -3dB.

Hence, On the eqn for M put $u = u_b$ and equating

$$\text{to } \frac{1}{\sqrt{2}}$$

$$M = \frac{1}{[(1-u)^2 + 4\zeta^2 u^2]^{\frac{1}{2}}}$$

'u' is Replaced with u_b

$$\frac{1}{\sqrt{2}} = \frac{1}{[(1-u_b)^2 + 4\zeta^2 u_b^2]^{\frac{1}{2}}}$$

$$Y_2 = \frac{1}{(1 - u_b^2) + 4\zeta^2 u_b^2}$$

$$(1 - u_b^2) + 4\zeta^2 u_b^2 = 2$$

(LP)

$$1+4\zeta^2 - 2\zeta + 4\zeta^2 b^2 = 2.$$

$$4b^2 + 2\zeta b^2 (2\zeta - 1) - 1 = 0$$

$$4b^2 = x$$

$$x^2 + 2(2\zeta - 1)x - 1 = 0$$

$$x_1, x_2 = \frac{-2(2\zeta - 1) \pm \sqrt{4(2\zeta - 1)^2 + 4}}{2}$$

$$x_1 x_2 = \frac{2(1 - 2\zeta^2) \pm 2\sqrt{(1 \pm 4\zeta^2 - 4\zeta^2)} - 1}{2}$$

let us take only the positive sign.

$$x_1 = 1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

$$4b^2 = x$$

$$4b = \sqrt{x}$$

$$4b = \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

$$4b = \frac{\omega_b}{\omega_n}$$

$$\omega_b = \omega_n \omega_b$$

$$\text{Band width } \omega_b = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

Phase Margin :-

The open loop transfer fun of second order system is given by.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

The sinusoidal transfer fun $G(j\omega)$ is obtained by letting $s = j\omega$

$$G(j\omega) = \frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)}$$

$$= \frac{\omega n^2}{\omega n^2 \left[\frac{-\omega}{\omega n^2} + j2\zeta \frac{\omega}{\omega n} \right]}$$

$$G(j\omega) = \frac{1}{-\omega^2 j 2\zeta u} = \frac{1}{j u (\zeta^2 + j u)}$$

$$M = |G(j\omega)| = \frac{1}{\sqrt{u^4 + 4\zeta^2 u^2}}$$

Phase angle of $G(j\omega)$ is

$$\alpha = \angle G(j\omega) = -90^\circ - \tan^{-1} \left(\frac{u}{2\zeta} \right)$$

$$\alpha = \angle G(j\omega) = -90^\circ - \tan^{-1} \left(\frac{u}{2\zeta} \right)$$

at the gain cross over frequency ω_{gc} , the magnitude of $G(j\omega)$ is unity.

Let normalised Gain cross over frequency

$$u_{gc} = \frac{\omega_{gc}}{\omega}$$

On substituting "u" by u_{gc} in the magnitude eqn and equating to unity-

$$\therefore \text{at } u = u_{gc}, |G(j\omega)| = \frac{1}{\sqrt{u_{gc}^4 + 4\zeta^2 u_{gc}^2}}$$

$$1 = \frac{1}{\sqrt{u_{gc}^4 + 4\zeta^2 u_{gc}^2}}$$

$$\left(\sqrt{u_{gc}^4 + 4\zeta^2 u_{gc}^2} \right)^2 = 1$$

$$u_{gc}^4 + 4\zeta^2 u_{gc}^2 = 1.$$

$$u_{gc}^4 + 4\zeta^2 u_{gc}^2 - 1 = 0.$$

$$u_{gc}^2 = x.$$

$$x^2 + 4\zeta^2 x - 1 = 0$$

$$-4\zeta^2 \pm \sqrt{(4\zeta^2)^2 + 4}$$

4.

$$= \frac{-4\zeta^2 \pm \sqrt{16\zeta^4 + 4}}{2}$$

$$x_1 x_2 = -2\zeta^2 \pm \sqrt{4\zeta^4 + 1}$$

$$w_{gc} = \sqrt{x}$$

$$w_{gc} = \sqrt{-2\zeta^2 \pm \sqrt{4\zeta^4 + 1}}$$

The phase margin $\gamma = 180^\circ + \angle G(j\omega)$ at $\omega = w_{gc}$, $U = w_{gc}$

$$\gamma = 180^\circ + (-90 - \tan^{-1} \frac{w_{gc}}{2\zeta})$$

$$= 180^\circ - (90 - \tan^{-1} \frac{-2\zeta^2 \pm \sqrt{4\zeta^4 + 1}}{2\zeta})$$

Bode plot :-

The frequency response plots are.

1. Bode plot
2. polar plots (or) Nyquist plot
3. Nichols plot
4. Mand N Circles
5. Nichols chart

1. Draw the ~~polar~~ Bode plot for the following transfer fun and determine the system gain K for the gain cross over frequency to be 5 rad/sec $G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)}$

Step - 1:-

The Sinsoidal T. f $G(j\omega)$ is obtained by repla - cing s by $j\omega$ in the given T.f.

$$s = j\omega$$

$$G(j\omega) = \frac{K(j\omega)^2}{[1+(0.2j\omega)][1+0.02(j\omega)]}$$

Let $k=1$

$$G(j\omega) = \frac{C(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

Step - 02 :-

Magnitude plot :-

The corner frequencies are $\omega_{c1} = \frac{1}{0.2}$

$$\Rightarrow \frac{1}{0.2} = 5 \text{ rad/sec}$$

$$\omega_{c2} = \frac{1}{T_g} = \frac{1}{0.02} = 50 \text{ rad/sec}$$

Term	Corner frequency rad/sec	slop dB/sec.	change in slope
$(j\omega)^2$		40dB	
$\frac{1}{(1+0.2j\omega)}$	$\omega_{c1} = 5 \text{ rad/sec}$	-20dB	$-40 + (-20)$ = 20dB
$\frac{1}{1+0.02j\omega}$	$\omega_{c2} = 50 \text{ rad/sec}$	-20dB	$20 + (-20) = 0 \text{ dB}$

Choose a low frequency ω_L such that $\omega_L < \omega_{c1}$ and choose a high-frequency ω_H such that $\omega_H > \omega_{c2}$

$$\therefore \omega_L = 0.5 \text{ rad/sec}$$

$$\omega_H = 100 \text{ rad/sec}$$

Let

$$A = (20G(j\omega)) \text{ in dB}$$

Let us calculate "A" at $\omega_L, \omega_{c1}, \omega_{c2}, \omega_H$ when

$$\omega = \omega_L$$

$$A = 20 \log |G(j\omega)|$$

unit \rightarrow Pg - 20%

$$\textcircled{2) } \quad -A = 20 \log |G(j\omega)| = 20 \log (0.5)^{-1} \\ = -12 \text{ dB}$$

when $\omega = \omega_c$, $A = 20 \log |G(j\omega_c)|$
 $= 20 \log |5^{-1}| = 28 \text{ dB}$

when $\omega = \omega_{c_2}$,

$\left[\text{slope from } \omega_c \text{ to } \omega_{c_2} \log \frac{\omega_{c_2}}{\omega_c} \right] + A \text{ at } \omega = \omega_c$

$$-A = 20 \log \frac{50}{5} + 28 \text{ dB}$$

$$-A = 20 \log 10 + 28 \text{ dB.}$$

$$\boxed{-A = 48 \text{ dB.}}$$

when $\omega = \omega_h$

$-A = \left[\text{slope from } \omega_{c_2} \text{ to } \omega_h \log \frac{\omega_h}{\omega_{c_2}} \right] + A \text{ at } \omega = \omega_{c_2}$

$$-A = \left[20 \log \frac{100}{50} \right] + 48$$

$$= 48 \text{ dB}$$

let the points A, B, C, D be the points according to frequencies $\omega_L, \omega_C, \omega_{C_2}$ and ω_h respectively of the magnitude.

In a semilog graph sheet choose a small scale

for the points A, B, C, & D on the graph

join the points by straight lines and mark the slope on the respective regions.

Step-3:-

Phase plot :-

The phase angle $G(j\omega)$ as a function of ω is given by

$$G(j\omega) = \frac{(j\omega)^2}{(1 + (0.2)j\omega)(1 + 0.02j\omega)}$$

$$= 180^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega)$$

ω rad/sec (degree)	$\tan^{-1}(0.2)$ (degree)	$\tan^{-1}(0.02\omega)$ (degree)	ϕ (degree)
0.5	5.7	0.6	174
1	11.3	1.1	168
5	45	5.7	130
10	68.4	11.3	106
50	84.3	45	50
100	87.1	63.4	30

Step - 4 :-

Calculation of k :-

Given that the gain cross over frequency is 5 rad/sec at $\omega = 5$ rad/sec, the gain is 28 db.

If gain cross over frequency is 5 rad/sec then at that frequency the db gain should be zero

Hence to every point of magnitude plot Adb gain of -28 db should be added. the addition of -28 db shift the plot down-words..

$$\therefore 20 \log k = -28 \text{ db}$$

$$\log k = -\frac{28}{20}$$

$$k = 10^{-\frac{28}{20}}$$

$$k = 0.0398$$

* Draw the Bode plot for the following transfer fun and determine phase margin and gain margin

$$G(s) = \frac{75(1+0.25)}{s(j\omega + 165 + 0)}$$

Step-1:-

G.T

$$G(s) = \frac{75(1+0.25)}{s(s^2 + 165 + 100)}$$

$$= \frac{75(1+0.25)}{s \cdot 100 \left(\frac{s^2}{100} + \frac{165}{100} + 1 \right)}$$

$$G(s) = \frac{0.75(1+0.25)}{s(1+0.01s^2 + 0.165)}$$

s is Replaced by $j\omega$.

$$\begin{aligned} G(j\omega) &= \frac{0.75(1+0.2(j\omega))}{j\omega[1+(j\omega)^2 0.01 + 0.165(j\omega)]} \\ &= \frac{0.75[1+j0.2\omega]}{j\omega[1-0.01\omega^2 + j0.16\omega]} \end{aligned}$$

Step-2:-

Magnitude plot :-

$$\omega_c = \frac{1}{T} = \frac{1}{0.2} = 5 \text{ rad/sec}$$

$$\tilde{\zeta}^2 + 2\tilde{\xi}\tilde{\omega}_n + \tilde{\omega_n}^2 = s^2 + 165 + 100$$

$$\tilde{\omega_n}^2 = 100$$

$$\omega_{c_1} = \tilde{\omega_n} = 10.$$

The Corner frequencies are

$$\omega_{c_1} = 5 \text{ rad/sec}$$

$$\omega_{c_2} = 10 \text{ rad/sec.}$$

Term	Corner frequency (rad/sec)	Slope (db)	Change in slope.
$\frac{0.75}{j\omega}$		-20	0
$1 + j8.2\omega$	$\omega_{C_1} = \pi \text{ rad/sec}$	20	$-20 + 20 = 0$
$\frac{1}{1 - 0.01\omega^2}$	$\omega_{C_2} = 10 \text{ rad/sec}$	-40	$0 + (-40) = -40$

Chose a low frequency ω_L such that $\omega_L < \omega_{C_1}$ and chose a high frequency ω_H such that $\omega_H > \omega_{C_2}$.

let $\omega_L = 0.5 \text{ rad/sec}$

$\omega_H = 20 \text{ rad/sec}$

$$\text{Gain} = \frac{1}{s^2 + \frac{1}{\omega_L^2}} = \frac{1}{s^2 + \frac{1}{0.25}} = \frac{1}{s^2 + 4}$$

$$\text{Gain} = \frac{1}{s^2 + 4} = \frac{1}{s^2 + j2s + 4}$$

$$\text{Gain} = \frac{1}{s^2 + j2s + 4}$$

$$\text{Gain} = \frac{1}{s^2 + j2s + 4} = \frac{1}{s^2 + j2s + 4}$$

$$\text{Gain} = \frac{1}{s^2 + j2s + 4} = \frac{1}{s^2 + j2s + 4}$$

$$\text{Gain} = \frac{1}{s^2 + j2s + 4} = \frac{1}{s^2 + j2s + 4}$$

(25) Let $A = |G(j\omega)| \text{ in db}$
 let us calculate A at $\omega_L, \omega_C, \omega_{C_2}, \omega_h$ when

$$\omega = \omega_L$$

$$A = 20 \log |G(j\omega)| = 20 \log \left| \frac{0.75}{j\omega} \right|$$

$$= 20 \log \left| \frac{0.75}{0.5} \right|$$

$$A = 3.5 \text{ db}$$

when

$$\omega = \omega_C, A = 20 \log |G(j\omega)| = 20 \log \left| \frac{0.75}{5} \right| = -16.5 \text{ db}$$

when

$$\omega = \omega_{C_2}$$

$$A = \left[\text{slope from } \omega_C \text{ to } \omega_{C_2} \times \log \frac{\omega_{C_2}}{\omega_C} \right] + A \text{ at } \omega = \omega_C$$

$$A = \left[0 \times \log \frac{10}{5} \right] - 16.5 = -16.5 \text{ db}$$

when $\omega = \omega_h$

$$A = \left[\text{slope from } \omega_{C_2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{C_2}} \right] + A \text{ at } \omega = \omega_{C_2}$$

$$A = \left[-40 \log \frac{80}{10} \right] - 16.5 = -28.5 \text{ db}$$

let the points A, B, C, D be the points corresponding to frequencies $\omega_L, \omega_C, \omega_{C_2}$ and ω_h respectively on the mag-magnitude plot

In a Semilog graph sheet choose a small scale fix the points $A, B, C, \& D$ on the graph
 join the points by straight lines and mark the slope on the respective regions.

Step-8:-

Phase plot :-

The phase angle $G(j\omega)$ as function of ω is given by $\phi = \angle G(j\omega)$

$$G(j\omega) = \frac{0.75(1+j0.2\omega)}{j\omega(1-0.01\omega^2+j0.16\omega)}$$

$$\phi = \underline{\text{Im}(j\omega)} = \tan^{-1}(0.2\omega) - 90^\circ - \tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega}\right)$$

$\omega \text{ rad/sec}$	$\tan^{-1}(0.2\omega) \text{ (degree)}$	$\tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega}\right) \text{ (degree)}$	$\phi \text{ degree}$
0.5	5.7	4.6	-89
1	11.36	9.2	-88
5	45	46.8	-92.
10	63.4	90	-116
20	75.9	-46.8 + 180 = 133.2	-118
50	84.3	161.5	-168
100	87.13	170.8	-174.

Gain margin = ∞

3. Given $G(s) = \frac{Ke^{-0.25}}{s(s+2)(s+8)}$. find K, so that the system is stable with

(a) Gain margin = 2dB

(b) Phase angle = 45°

let us consider $K=1$, and convert the given transfer fun to time constant form or Bode form

$$G(s) = \frac{K e^{-0.25}}{s \cdot 2(1+0.5s)(1+0.125s)}$$

$$G(s) = \frac{0.625 e^{-0.25}}{s(1+0.5s)(1+0.125s)}$$

The sinusoidal transfer-fun $G(j\omega)$ is obtained by replacing s by $j\omega$ in $G(s)$

$$G(j\omega) = \frac{0.0625 e^{j0.25\omega}}{(j\omega(1+0.5j\omega)(1+0.125j\omega))}$$

Note :-

$$0.0625 e^{-j0.2w}$$

$$\text{magnitude} = |0.0625 e^{-j0.2w}|$$

$$= 0.0625$$

$$\text{Angle } \underline{|0.0625 e^{-j0.2w}|}$$

$$= -0.2w \text{ rad.}$$

Step-2:-

Magnitude plot :-

The Corner-frequencies are $\omega_{C_1} = \frac{1}{T_1} = \frac{1}{0.5} = 2 \text{ rad/sec.}$

$$\omega_{C_2} = \frac{1}{T_2} = \frac{1}{0.125} = 8 \text{ rad/sec.}$$

Terms	Corner-frequency	Slope(dB)	Change Slope(dB)
$\frac{0.0625}{j\omega}$	-	-20	-
$\frac{1}{1+j0.5\omega}$	$\omega_q = 2$	-20dB	$-20 + (-20) = -40 \text{ dB}$
$\frac{1}{1+j0.125\omega}$	$\omega_{q_2} = 8 \text{ rad/sec}$	-20dB	$-40 - 20 = -60 \text{ dB.}$

choose a low frequency ω_L such that $\omega_L < \omega_{C_1}$ and choose a high frequency ω_H such that $\omega_H > \omega_{C_2}$.

let

$$\omega_L = 0.5 \text{ rad/sec.}$$

$$\omega_H = 50 \text{ rad/sec}$$

let

$$-A = (g(C(\omega))) \text{ in dB}$$

let us calculate A at $\omega_L, \omega_C, \omega_{C_2}, \omega_H$

when

$$\omega = \omega_L, \text{ magnitude } A = 20 \log |C(\omega)|.$$

$$A = 20 \log \left| \frac{0.0625}{j\omega} \right|$$

unit - 30 Pg - 267/42

$$= 20 \log \left| \frac{0.0625}{0.5} \right|$$

$$A = -18.06 \text{ dB}$$

when

$$\omega = \omega_{C_1}, \text{ magnitude } A = 20 \log \left| \frac{0.0625}{\omega} \right|.$$

$$A = -30.10 \text{ dB}$$

when

$$\omega = \omega_{C_2}, A = [\text{slope from } \omega_{C_1} \text{ to } \omega_{C_2} \log \left(\frac{\omega_{C_2}}{\omega_{C_1}} \right)] + A + \omega_{C_1}$$

$$= -40 \log \left(\frac{8}{2} \right) + (-30.1)$$

$$= -54 \text{ dB}$$

when,

$$\omega = \omega_n, A = [\text{slope from } \omega_{C_2} \text{ to } \omega_n \log \left(\frac{\omega_n}{\omega_{C_2}} \right)] + A + \omega_{C_2}$$

$$A = -60 \log \left(\frac{54}{8} \right) - 54.$$

$$A = -101.7 \text{ dB}$$

When the points A, B, C & D be the points corresponding to frequencies $\omega_1, \omega_{C_1}, \omega_{C_2}, \omega_n$ respectively on the magnitude plot in a semilog graph sheet. Find the points by straight lines and mark the slope on the respective region

Step-3:-

The phase angle of $G(j\omega)$ as fun of ω is given

by

$$\theta = \angle G(j\omega) = -90^\circ - \tan^{-1}(0.5\omega) = \tan^{-1}(0.0125\omega) - 0.2\omega \times \frac{180^\circ}{\pi}$$

ω rad/sec	$0.2\omega \times \frac{180^\circ}{\pi}$	$\tan^{-1}(0.5\omega)$	$\tan^{-1}(0.125\omega)$	ϕ
0.01	-0.11	0.286	0.071	-90.46
0.1	-1.14	2.86	0.416	-99.7
0.5	-5.72	14.03	8.57	-113.3
1	-11.45	26.56	7.125	-135.1
2	-22.67	45	14.03	-171.9
3	-34.37	56.36	20.55	-201.2
4	-45.83	63.43	26.56	-225.8
8	-91.67	75.96	45	-302.8
9	-572.95	87.90	80.90	-831.3

Step - IV :-

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Calculation of K :-

$$\text{Phase margin } \gamma = 180^\circ + \phi_{gc}$$

when

$$\gamma = 45^\circ$$

$$45^\circ = 180^\circ + \phi_{gc}$$

$$\phi_{gc} = -180^\circ + 45^\circ$$

$$\phi_{gc} = -135^\circ$$

With $K=1$, the dB gain at $\phi = -135^\circ$ is -29dB . This gain should be made zero to have $+24\text{dB}$ added.

$$20 \log K = 24$$

$$\log K = 24/20$$

$$K = 10^{24/20}$$

$$K = 105.84$$

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with $k=1$, the gain margin $-G(s_2) = 3 \text{dB}$ but the new desired gain margin is 2dB hence to every plot of magnitude plot a dB gain of ~~at~~ 3dB should be added.

Procedure for mag. Bode plot :-

Procedure for magnitude plot of Bode plot :-

$$G(s) = \frac{k(1+sT_1)^{-2}}{s^2(1+sT_2)(1+sT_3)}$$

Put $s = j\omega$.

$$G(j\omega) = \frac{k(1+j\omega T_1)^{-2}}{(j\omega)^2(1+j\omega T_2)(1+j\omega T_3)}$$

$$T_2 < T_3 < T_1$$

Corner frequencies are.

$$\omega_{C_1} = \frac{1}{T_1}, \quad \omega_{C_2} = \frac{1}{T_2}, \quad \omega_{C_3} = \frac{1}{T_3}$$

The magnitude of very first and higher order terms are added one by one in the increasing order of the corner frequency.

Step-1:-

Convert the transfer function into Bode form (or time constant form). The Bode form of the transfer function is.

$$G(s) = \frac{k(1+sT_1)}{s(s+T_2)(1+\frac{s}{\omega_n} + 2\zeta\frac{s}{\omega_n})}$$

Putting $s = j\omega$.

$$G(j\omega) = \frac{k(1+j\omega T_1)}{(j\omega)(1+j\omega T_2)\left(1 - \frac{j\omega}{\omega_n} + 2\zeta\frac{j\omega}{\omega_n}\right)}$$

Step-2:-

List the corner frequencies in the increasing order and prepare a table as shown below.

Term	Corner frequency (rad/sec)	slope (db/deg)	Change in slope (db/deg).

In the above table Enter k or $k(j\omega)^n$ (or) $\frac{1}{k(j\omega)^n}$ as the first term then other terms in the increasing order of corner frequencies. Then enter the corner frequency slope by each term and change in slope at every corner frequency.

Step-3:-

Choose an arbitrary frequency i.e., low frequency ω_L which is lesser than the lowest corner frequency. Calculate the db magnitude of k (or) $k(j\omega)^n$ (or) $\frac{1}{k(j\omega)^n}$ at ω_L and at the lowest corner frequency.

Step-4:-

Then calculate the gain at every corner frequency one by one by using the formula.

Gain at ω_y = Change in gain from ω_L to ω_y + Gain at ω_L

$$= \left(\text{slope from } \omega_L \text{ to } \omega_y \times \log \frac{\omega_y}{\omega_L} \right) + \text{Gain at } \omega_L.$$

Step-5:-

Choose a high frequency ω_H which is greater than the highest corner frequency. Calculate the gain at ω_H by using the formula at step-4.

Step-6:-

In a Semilog Graph sheet mark the required Range of frequency on x-axis and Range on db magnitude on y-axis after choosing proper unit.

Step-7:-

Mark all the points obtained in steps 3, 4, 5 on the graph and join the points by straight lines, mark the slope at every part of the graph.

Procedure for phase plot of Bode plot:-

The phase plot is an exact and no approximations are made while drawing the phase plot hence the exact angles of $G(j\omega)$ are computed for various values of ' ω ' and tabulated.

The choice of frequencies are preferred the frequencies chosen for magnitude plot, usually the magnitude plot and phase plot are drawn in a single semilog graph sheet on a common frequency scale.

Take another y-axis in the graph where the magnitude plot is drawn and in this y-axis mark the desired range of phase angles after choosing proper unit.

From the tabulated values of ω and angles, mark all the points on graph, join the points on a smooth curve.

Determination of gain margin, phase margin.

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from Bode plot

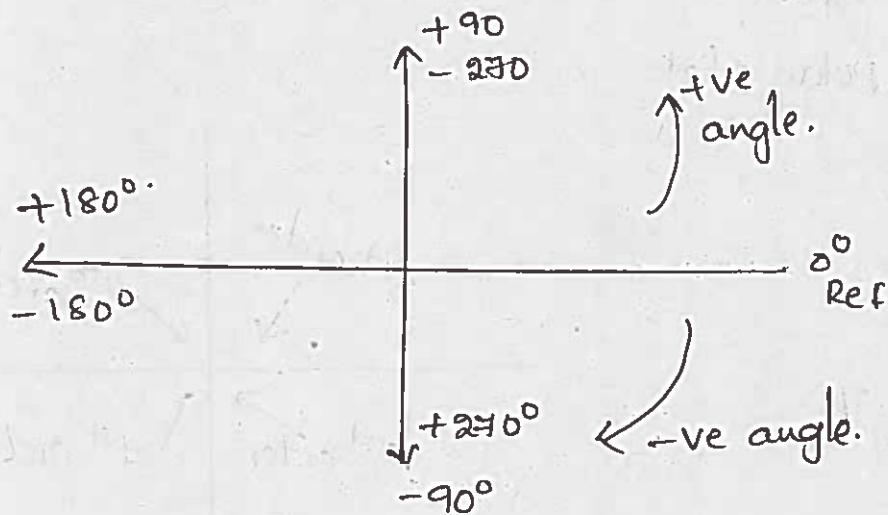
The gain margin in dB is given by the negative of dB magnitude of $G(j\omega)$ at the phase cross over frequency ω_{pc} . The ω_{pc} is the frequency at which phase or $G(j\omega)$ is -180° . If the dB magnitude of $G(j\omega)$ at ω_{pc} is negative the gain margin is positive and vice versa.

Let ϕ_{qc} be the phase angle of $G(j\omega)$ at gain cross over frequency ω_{qc} , the ω_{qc} is the frequency at which the dB magnitude of $G(j\omega)$ is zero. Now the phase margin γ is given by $\gamma = 180^\circ + \phi_{qc}$. If ϕ_{qc} is less negative from -180° then phase margin is positive and vice versa.

* stability Analysis in frequency domains:-

Polar plot:-

The polar plot of a sinusoidal t.f $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity.



Alternatively, if $G(j\omega)$ can be expressed in rectangular coordinates as.

$$G(j\omega) = G_R(j\omega) + jG_I(j\omega)$$

$G_R(j\omega)$ = Real part part of $G(j\omega)$

$G_I(j\omega)$ = imaginary part of $G(j\omega)$

It varies from $G_R(j\omega)$ & $G_I(j\omega)$ as ω is varied from 0 to ∞ .

Note :-

For minimum phase t.f which has only poles the type no. of the type determines at what quadrant the polar plot starts and the order of the system determines at what quadrant the polar plot ends.

\downarrow | \downarrow Type - 8

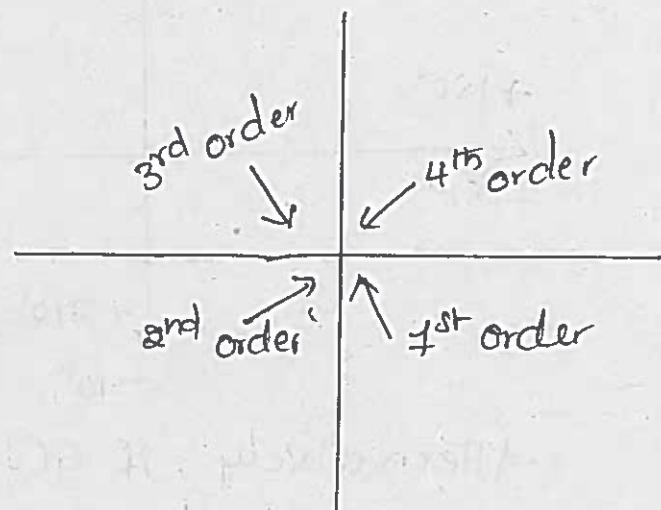
$\overrightarrow{\text{type - 2}}$
 $\overrightarrow{\text{}}$

type - 0

\uparrow | 1

Type - 1.

Start of Polar plot



Determination of gain margin & phase margin from polar plot

The Gain Margin is defined as the inverse of the magnitude of $G(j\omega)$ at phase cross over frequency.

The phase cross over frequency is the frequency at which the phase of $G(j\omega)$ is 180° .

Now the gain margin $K_g = \frac{1}{|G_B|}$

$K_g = \frac{1}{|G_B|}$ where G_B is the magnitude

The phase margin defined as phase margin $\gamma = 180 + \phi_{gc}$ where ϕ_{gc} is the phase angle of $G(j\omega)$ at given cross over frequency.

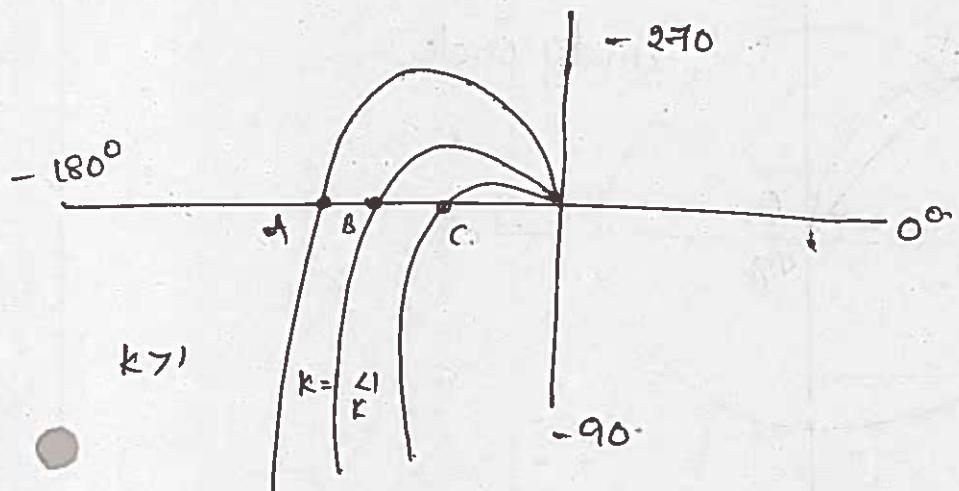
The gain cross over frequency is the frequency at which the magnitude of $G(j\omega)$ is unity.

Gain adjustment using Polar plot:-

To determine K for Specified (G_m)

Draw $G(j\omega)$ loci with $K=1$ let it cut the -180° axis at point B Corresponding to gain of G_B .

Let the specified gain margin be α db for this gain margin the $G(j\omega)$ locus will cut -180° at point A whose magnitude is G_A .

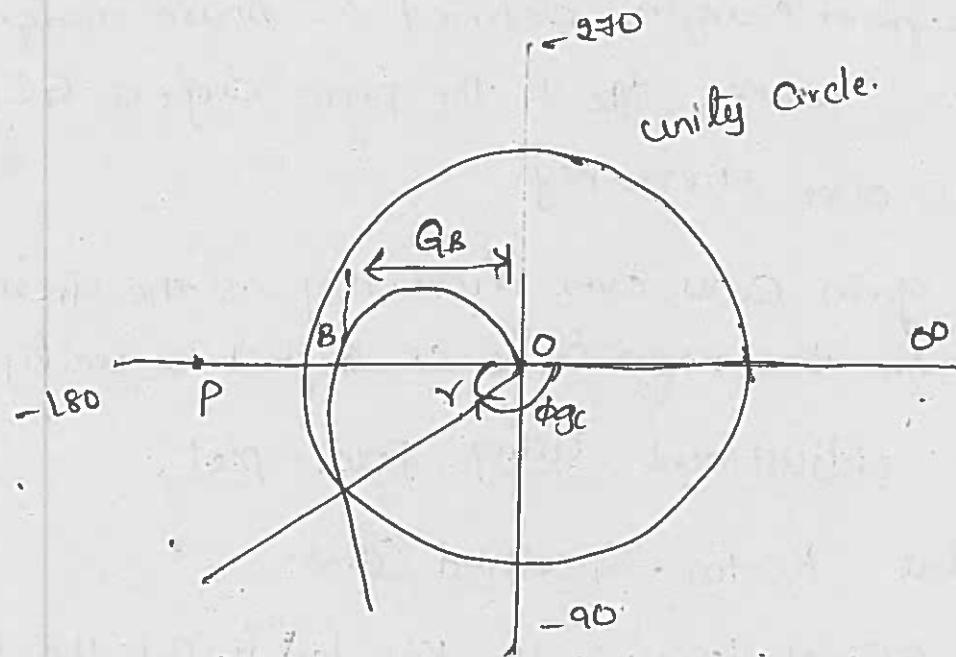


$$20 \log \frac{1}{G_A} = \alpha.$$

$$\log \frac{1}{G_A} = \frac{\alpha}{20}$$

$$\frac{1}{G_A} = 10^{\alpha/20}$$

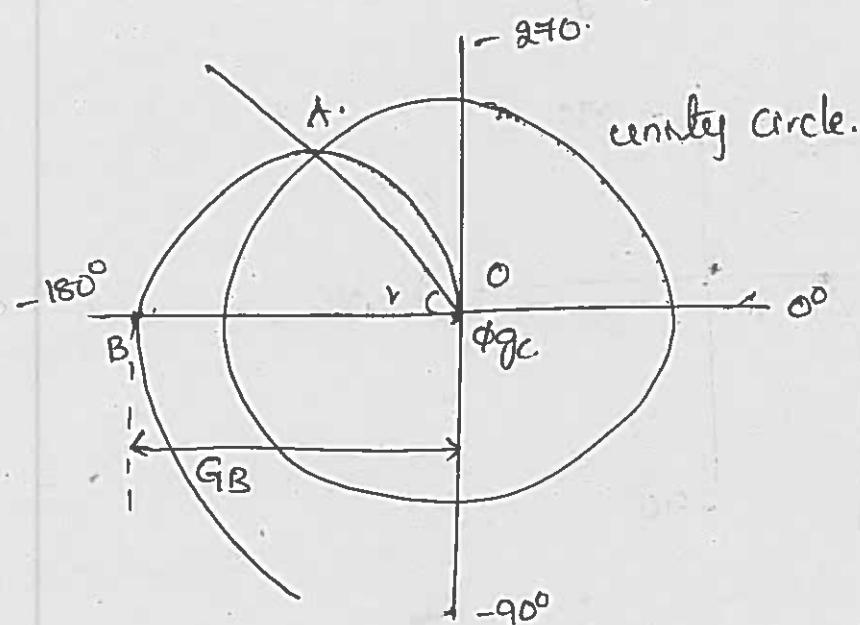
$$G_A = \frac{1}{10^{\alpha/20}}$$



$$\text{Gain margin } K_g = \frac{1}{G_B}$$

$$\text{Phase margin } \gamma = 180^\circ + \phi_{Gc}.$$

The above fig shown +ve gain margin and phase margin



$$\text{Gain margin } K_g = \frac{1}{G_B}$$

$$\text{Phase margin } \gamma = 180 + \phi_{Gc}$$

The above fig shown -ve gain margin and phase margin.

Nyquist plots :-

The polar plot is the locus of vectors $|G(j\omega)| < G(j\omega)$ as ω is varied from zero to infinity.

The polar plot is called Nyquist plot.

let us consider closed loop $t = f$.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s) H(s)}$$

$$1 + G(s) H(s) = 0$$

$$\text{Let } F(s) = 1 + G(s) H(s).$$

$$G(s) H(s) = \frac{k (s+z_1)(s+z_2) \dots (s+z_m)}{(s+p_1)(s+p_2) \dots (s+p_n)}$$

where $m \leq n$

$$F(s) = 1 + G(s) H(s)$$

$$= 1 + \frac{k(z_1+s)(z_2+s) \dots (z_m+s)}{(s+p_1)(s+p_2) \dots (s+p_n)}$$

$$= \frac{(s+p_1)(s+p_2) \dots (s+p_n) + k(z_1+s)(z_2+s) \dots (z_m+s)}{(s+p_1)(s+p_2) \dots (s+p_n)}$$

$$= \frac{(s+z'_1)(s+z'_2) \dots (s+z'_n)}{(s+p_1)(s+p_2) \dots (s+p_n)}$$

where z'_1, z'_2, \dots, z'_n are zeros of $f(s)$.

$$f(s) = 0$$

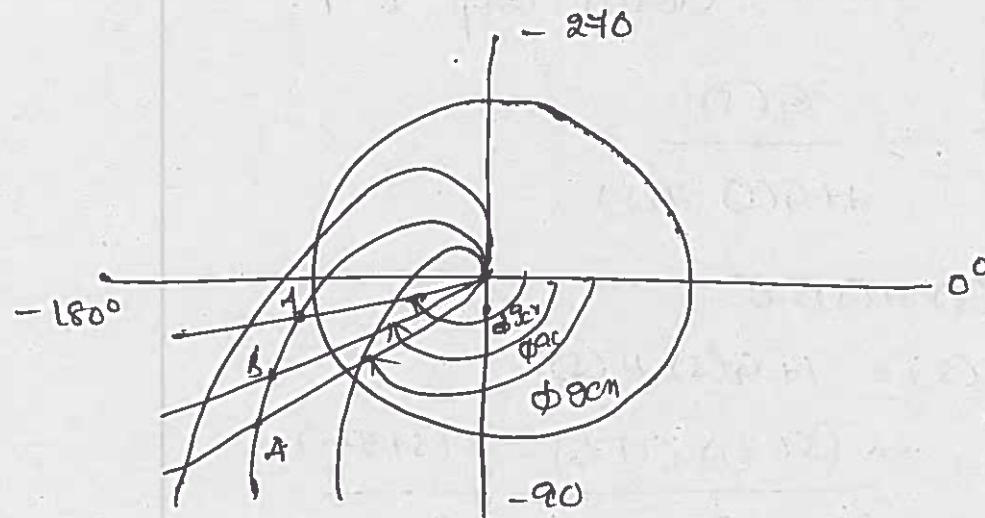
$$(s+z'_1)(s+z'_2) \dots (s+z'_n) = 0$$

$$K = \frac{G_A}{G_B}$$

If $K > 1$, the gain is increased.

$K < 1$, the gain is decreased.

To determine K for specified Pin:-



$$\chi^0 = 180^\circ + \phi_{G(j\omega)}$$

$$\phi_{G(j\omega)} = \chi^0 - 180^\circ$$

In the polar plot, the radial line corresponding to $\phi_{G(j\omega)}$ will cut the locus of $G(j\omega)$ with $K=1$. at point 1 and the magnitude corresponding to the point to G_A .

Now

$$K = \frac{G_B}{G_A}$$

$$K = \frac{1}{G_A}$$

where.

$$G_B = 1$$

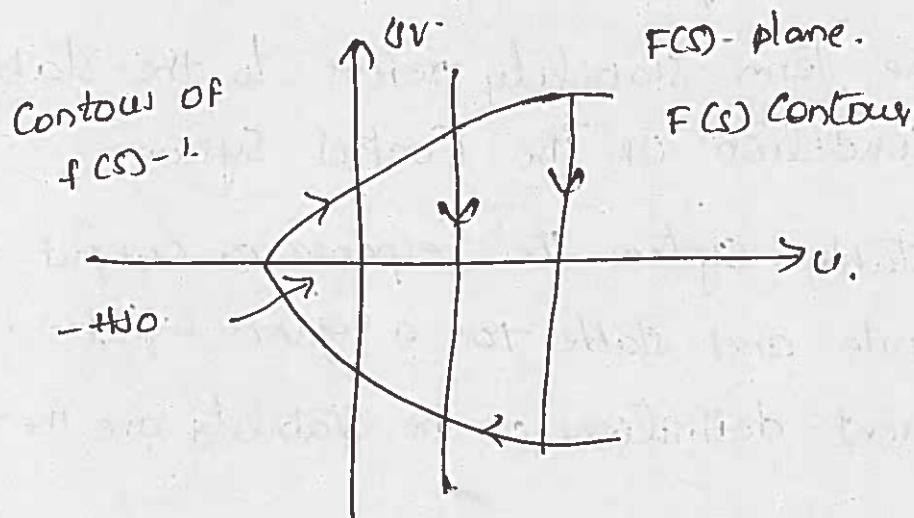
$$N = P - Z$$

when $Z=0$, $N=P$

where $Z \neq 0$, $N \neq P$

$$G(s) + H(s) = [1 + G(s)H(s)]^{-1}$$

$$= F(s) - 1$$



Thus the encirclement of the origin of $F(s)$ -plane by the $F(s)$ contour is equivalent to the encirclement of the point $(-1+j0)$ by the $G(s)+H(s)$ contour.

The Nyquist stability criterion has been proposed based on this concept

The Nyquist Contour is directed clockwise and Comprises of two segments.

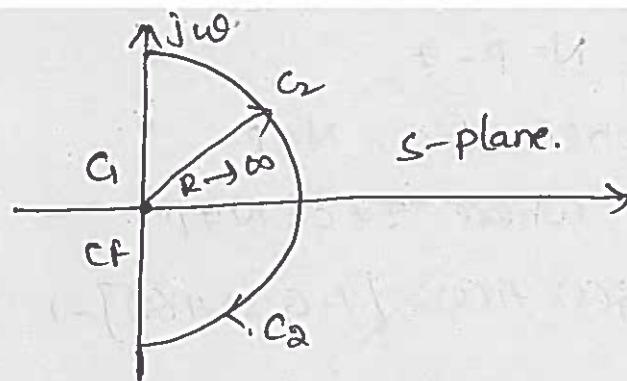
(i) An infinite line segment C_1 along the imaginary axis

(ii) An arc, C_2 at infinite radius.

along C_1 , $s=j\omega$ it varying from $-j\infty$ to $+j\infty$

along C_2 , $s=1t R e^{j\theta}$ with varying from $R \rightarrow \infty$

$\theta \rightarrow \pi/2$ to $-\pi/2$.



Stability analysis:-

The term stability refers to the stable working condition of the Control System.

In a stable system the response or output is predictable finite and stable for a given input.

The different definitions of the stability are the following.

1. A system is stable if its o/p is bounded for any bounded (finite) input
2. A system is asymptotically stable if in the absence of the i/p the o/p tends towards zero irrespective of initial conditions
3. The system is stable if for a bounded disturbing input signal the o/p vanishes ultimately as approaches infinity.
4. A system is unstable if for a bounded disturbing input signal the o/p is infinite amplitude or oscillatory
5. For a bounded input signal, if the output has constant amplitude oscillations then system may be stable or unstable under some limited constraints such a system is called limitedly stable.

Static error constant: UNIT-IV
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 Within a C.S is exerted with standard i/p signal. The steady state error may be zero, or constant or infinity. The value of steady state error depends on the type number and i/p signal.

Type zero system will have constant zero error when the i/p is unit step signal.

error Type one system will have constant steady state when the i/p is ramp signal.

Type two system will have a constant steady state error when the i/p is parabolic.

Positional error signal (or) Constant (k_p)

$$k_p = \lim_{s \rightarrow 0} G(s) H(s)$$

$$\text{Velocity error constant } k_v = \lim_{s \rightarrow 0} s G(s) H(s)$$

$$\text{acceleration error constant } k_a = \lim_{s \rightarrow 0} s^2 G(s) H(s)$$

where k_p, k_v, k_a are in general called static error constant.

Steady state error when the i/p is unit step signal:

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot R(s)}{1 + G(s) H(s)}$$

$$r(t) = 1$$

$$R(s) = \frac{1}{s}$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot \frac{1}{s}}{1 + G(s) H(s)} = \lim_{s \rightarrow 0} \frac{1}{1 + G(s) H(s)}$$

$$e_{ss} = \frac{1}{1 + \lim_{s \rightarrow 0} G(s) H(s)}$$

$$K_p = \lim_{s \rightarrow 0} G(s) H(s)$$

$$ess = \frac{1}{1 + K_p}$$

For type - 0 system:-

$$K_p = \lim_{s \rightarrow 0} G(s) H(s) = \lim_{s \rightarrow 0} \frac{K(s+z_1) \dots (s+z_n)}{s^N (s+p_1) \dots (s+p_n)}$$

$$N=0$$

$$= \lim_{s \rightarrow 0} \frac{K \cdot z_1 z_2 \dots}{p_1 p_2 \dots}$$

$$= \text{constant}$$

$$\text{Steady state error } ess = \frac{1}{1 + K_p}$$

$$ess = \frac{1}{1 + \text{constant}}$$

$$ess = \text{constant}$$

When r_p is unit step there will be a constant steady state error.

For type - 1 system:-

$$K_p = \lim_{s \rightarrow 0} G(s) H(s) = \lim_{s \rightarrow 0} \frac{K(s+z_1) \dots (s+z_n)}{s^N (s+p_1) \dots (s+p_n)}$$

$$N=1$$

$$K_p = \frac{K(z_1 z_2 z_3)}{s(p_1 p_2 p_3)}$$

$$K_p = \infty$$

$$\text{Steady state error} = \frac{1}{1 + K_p} = \frac{1}{1 + \infty}$$

$$A=1, B=\frac{1}{3}, C=-\frac{4}{3}$$

$$C(s) = \frac{1}{s} + \frac{\frac{1}{3}}{s+1} - \frac{\frac{4}{3}}{s+4}$$

$$c(t) = 1 + \frac{1}{3} e^{-t} - \frac{4}{3} e^{-4t}$$

$$c(t) = 1 + \frac{1}{3} [e^{-t} - 4e^{-4t}]$$

Type number of control system:-

The type number of C.S is specified for loop of $G(s)H(s)$. The no. of poles lying at origin decides the type no. of the system.

In general if N is the no. of poles at the origin then the type number is N.

$$G(s)H(s) = \frac{k(P(s))}{B(s)} = \frac{k(s+z_1)(s+z_2)\dots(s+z_n)}{s^N(s+p_1)(s+p_2)\dots(s+p_N)}$$

z_1, z_2, z_3, \dots pole zero's of the numerator.

p_1, p_2, p_3, \dots are poles of the denominator.

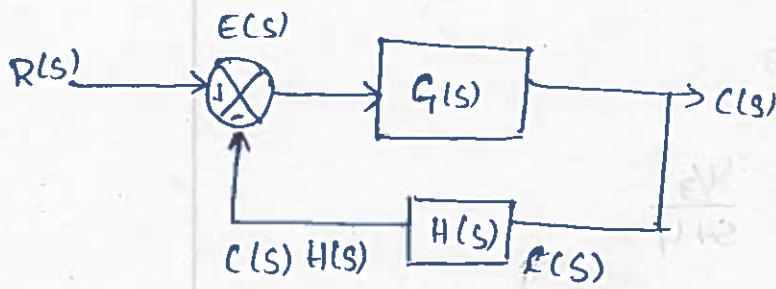
k = constant

N = no. of poles at origin.

Steady state error:-

The steady state error is the value of error signal $e(t)$ when $t \rightarrow \infty$. The steady state error is the measure of system's accuracy. These errors arise from the nature of I/P's and type of system and from non linearities of system components.

The steady state performance of a stable control system is generally measured its steady state error to ramp, step, impulse and parabolic I/P's.



$$E(s) = R(s) - C(s)H(s)$$

$$C(s) = G(s)E(s)$$

$$E(s) = R(s) - G(s)E(s)H(s)$$

$$E(s)[1 + G(s)H(s)] = R(s)$$

$$E(s) = \frac{R(s)}{[1 + G(s)H(s)]}$$

$e(t)$ = error signal in time domain

$$e(t) = L^{-1} \left[\frac{R(s)}{1 + G(s)H(s)} \right]$$

e_{ss} = steady state error.

The steady state error is defined as the value of $e(t)$ when $t \rightarrow \infty$

$$e_{ss} = \lim_{t \rightarrow \infty} e(t)$$

$$F(s) = L[f(t)]$$

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

$$e_{ss} = \lim_{t \rightarrow \infty} e(t)$$

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} s \cdot \frac{R(s)}{1 + G(s)H(s)}$$

Hence in type 1-system when the i/p is unit step signal there will be a zero steady state error.

For type -2 system:-

$$k_p = \lim_{s \rightarrow 0} G(s) H(s) = \frac{K_1 z_1 z_2 \dots}{0(p_1, p_2, \dots)}$$

$$\text{steady state error } e_{ss} = \frac{1}{1+k_p} = \frac{1}{1+\infty}$$

$$e_{ss} = 0$$

Steady state error when the i/p is ramp signal:-

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot R(s)}{1+G(s)H(s)}$$

$$r(t) = t$$

$$R(s) = \frac{1}{s^2}$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \cdot \frac{1}{s^2}}{1+G(s)H(s)} = \lim_{s \rightarrow 0} \frac{\frac{1}{s}}{s(1+G(s)H(s))}$$

$$= \frac{\lim_{s \rightarrow 0} s + \lim_{s \rightarrow 0} s G(s)H(s)}{s^2 G(s)H(s)}$$

$$= \frac{1}{\lim_{s \rightarrow 0} s G(s)H(s)}$$

$$= \frac{1}{K_v}$$

$$e_{ss} = \frac{1}{K_v}$$

For type -0 systems:-

$$K_v = \lim_{s \rightarrow 0} s G(s) H(s) = \lim_{s \rightarrow 0} \frac{s K (s+z_1) \dots}{s^0 (s+p_1) \dots}$$

$$= 0$$

$$\text{Steady state error } e_{ss} = \frac{1}{K_v} = \frac{1}{0}$$

$$e_{ss} = \infty$$

Hence in type 3 zero system when the i/p is unit ramp signal there will be a infinity steady state error.

type - 1 system:-

$$K_v = \lim_{s \rightarrow 0} sG(s) + H(s)$$

$$= \lim_{s \rightarrow 0} \frac{s \cdot k(s+z_1) \dots}{s(s+p_1) \dots}$$

$$= \frac{k z_1 z_2 \dots}{p_1 p_2 \dots}$$

$$K_v = \text{constant}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\text{constant}}$$

$$e_{ss} = \text{constant.}$$

type - 2 system:-

$$K_v = \lim_{s \rightarrow 0} \frac{s \cdot k(s+z_1) \dots}{s^2(s+p_1) \dots}$$

$$= \frac{k z_1 z_2 \dots}{0(p_1 p_2 \dots)}$$

$$K_v = \infty$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

$$e_{ss} = 0$$

Hence in the system the type 2 system and above for the unit ramp signal the value of K_v is ∞ , so the steady state error is zero.

i) The steady state error when the R(s) is

$$\text{Where } R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$$

so for unity feedback system, $H(s) = 1$.

position error constant $K_p = \lim_{s \rightarrow 0} G(s)H(s)$.

$$\left[\frac{(s+2)s^2}{(s+2)(s+1)} \right] \frac{s}{s} \left[\frac{10(s+2)}{(s+2)(s+1)(s+2)s^2} \right] = \lim_{s \rightarrow 0} \frac{10(s+2)}{s^2(s+1)} = \frac{20}{0}$$

$\boxed{K_p = \infty}$

velocity error constant $K_v = \lim_{s \rightarrow 0} sG(s)H(s)$.

$$\left[\frac{(s+2)s^2}{(s+2)(s+1)(s+2)s^2} \right] \frac{s}{s} \left[\frac{10(s+2)}{(s+2)(s+1)(s+2)s^2} \right] = \lim_{s \rightarrow 0} \frac{10(s+2)}{s^2(s+1)} = \frac{20}{0}$$

$$\left[\frac{(s+2)s^2}{(s+2)(s+1)(s+2)s^2} \right] = \frac{20}{0} = \boxed{\alpha = K_v}$$

acceleration error constant $K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s)$

$$\left[\frac{(s+2)s^2}{(s+2)(s+1)(s+2)s^2} \right] + \left[\frac{(s+5)s^2}{(s+2)(s+1)(s+2)s^2} \right] = \lim_{s \rightarrow 0} s^2 \frac{10(s+2)}{s^2(s+1)}$$

$$= \frac{20}{0} \cdot \boxed{K_a = 20}$$

b) the error signal in s domain

$$e(s) = \frac{R(s)}{1+G(s)H(s)}$$

$$\text{Given that } G(s) = \frac{10(s+2)}{s^2(s+1)}$$

$$R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$$

$$H(s) = 1$$

$$\left(\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3} \right)$$

$$\frac{1}{1 + \left[\frac{10(s+2)}{s^2(s+1)} \right]} = 1$$

$$= \frac{\frac{3}{5} - \frac{2}{s^2} + \frac{1}{3s^2}}{s^2(s+1) + 10(s+2)}$$

$$= \frac{s^2(s+1) + 10(s+2)}{s^2(s+1)}$$

$$\epsilon(s) = \frac{3}{s} \left[\frac{s^2(s+1)}{s^2(s+1) + 10(s+2)} \right] - \frac{2}{s^2} \left[\frac{s^2(s+1)}{s^2(s+1) + 10(s+2)} \right] + \frac{1}{3s^2} \left[\frac{s^2(s+1)}{s^2(s+1) + 10(s+2)} \right].$$

The steady state error ϵ_{ss}

$$\epsilon_{ss} = \lim_{t \rightarrow \infty} \epsilon(t) = \lim_{s \rightarrow 0} s\epsilon(s).$$

$$\epsilon_{ss} = \lim_{s \rightarrow 0} \left[\frac{3s}{s^2} \left(\frac{s^2(s+1)}{s^2(s+1) + 10(s+2)} \right) \right] -$$

$$\left[\frac{2}{s^2} \frac{s^2(s+1)}{(s+1) + 10(s+2)} \right] + \left[\frac{s^2(s+1)}{s^2(s+1) + 10(s+2)} \right].$$

$$\epsilon_{ss} = 0 - 0 + \frac{1}{3 \times 20}$$

$$\boxed{\epsilon_{ss} = \frac{1}{60}}$$

for servomechanism with open loop transfer function

given below. Explain what type of op signals.

→ constant steady state errors and calculated this values.

$$G_1(s) = \frac{20(s+2)}{s(s+1)(s+3)}$$

Unit 4, Pg - 8/16

Let us assume unity system

$$\therefore H(s) = 1.$$

The open loop system has a pole at origin.
Hence it is a type 1 system.

In system with type 1 the ramp input will give the constant steady state error.

The steady state error with ramp input:

$$E_{SS} = \frac{1}{Kv} = \frac{1}{\lim_{s \rightarrow 0} s G(s) H(s)}.$$

$$E_{SS} = \lim_{s \rightarrow 0} \frac{1}{\frac{20(s+2)}{s(s+1)(s+3)}}.$$

$$= \frac{1}{\frac{20 \times 2}{1 \times 3}} = \frac{1}{40} = \frac{3}{40}.$$

$$E_{SS} = 0.075$$

Let us assume unity feedback system.

The open loop system has no pole at origin.

Hence it is type '0' system.

In system with type '0' the step input will give a constant steady state error

$$E_{SS} = \frac{1}{1 + K_p}.$$

$$K_p = \lim_{s \rightarrow 0} \frac{1}{s} \frac{G(s) + H(s)}{s}.$$

$$\lim_{s \rightarrow 0} \frac{1}{1+K_p} = \frac{1}{1+1.6} = 0.375.$$

$$G_L(s) = \frac{10}{s^2(s+1)(s+2)}.$$

let us assume unity feedback system $H(s) = 1$.
 The open loop system has no pole at origin.

let us assume unity feedback system '1'.
 The open loop system has two poles at origin.
 Hence it is a type '2'. the parabolic input will
 give constant steady state error.

$$E_{ss} = \frac{1}{K_a}$$

$$K_a = \lim_{s \rightarrow 0} \frac{1}{s^2 \cdot G_L(s) + H(s)}.$$

$$K_a \lim_{s \rightarrow 0} \frac{10}{s^2(s+1)(s+2)} = \frac{10}{2} = 5.$$

Controllers:

A controller is a device introduced in the system to modify the error signal and to produce a control signal. The mannering which the controller produces the control signal is called the control action.

There are proportional controller (P).

Integral controller (I)

Dervative controller (D)

1. Proportional controller + Integral controller (PI).
 2. Proportional + Derivative controller (PD).
 3. Proportional + Integral + Derivative (PID).

Proportional controller (P).

The 'P' controller is a device that produces a control signal $u(t)$ proportional to the error signal $e(t)$.

$\therefore P \text{ controller } u(t) \propto e(t)$

$$u(t) = k_p e(t).$$

Where

$k_p = \text{constant}$ (proportional gain).

On taking Laplace transform of above equation,

$\therefore \text{Transfer function } U(s) = k_p E(s)$

$$\frac{U(s)}{E(s)} = k_p$$

Integral controller:

It is a device that produces a control signal which is proportional to integral of the error signal.

Integral controller

$$= u(t) \propto \int e(t)$$

$$u(t) = k_i \int e(t)$$

By $k_i = \text{integral gain.}$

Laplace transform of above equation.

$$u(s) = K_p + \frac{e(s)}{s}$$

$$\frac{u(s)}{e(s)} = \frac{K_p}{s}$$

proportional + Integral controller (PI).

The proportional + integral controller produces an output signal consisting of two terms, one proportional to error signal and other proportional to integral of error signal.

$$\therefore \text{PI controller} = u(t) \alpha [e(t) + \int e(t) dt].$$

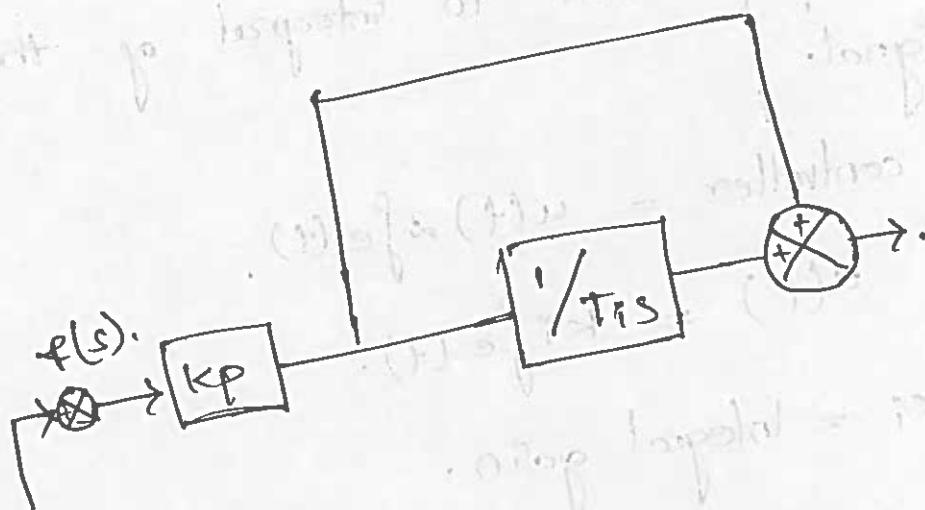
where K_p = proportional gain. (constant).

On taking Laplace transform of above equation:

$$u(s) = K_p \cdot t(s) + \frac{K_p}{T_i} \cdot \frac{e(s)}{s}$$

$$u(s) = K_p \left(1 + \frac{1}{T_i s} \right) e(s)$$

$$\frac{u(s)}{e(s)} = K_p \left(1 + \frac{1}{T_i s} \right)$$



Proportional + Derivative controller (PD)

13.

The PD controller produces an output signal consisting of two terms. They are

- ① Proportional error signal (and other).
- ② proportional to derivative of error signal.

Proportional + Derivative + Integral controller (PID)

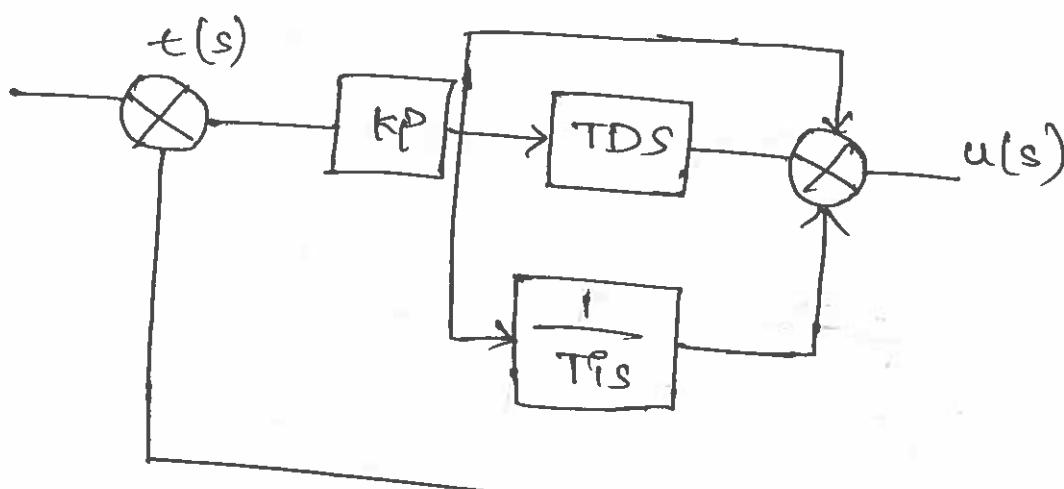
$$u(t) \propto [e(t) + \int e(t) dt + \frac{d}{dt} e(t)].$$

$$u(t) = k_p e(t) + \frac{k_p}{\tau_i} \int e(t) dt + k_p T_d \frac{d}{dt} e(t).$$

$$u(s) = k_p e(s) + \frac{k_p}{\tau_i} \frac{e(s)}{s} + k_p T_d s e(s).$$

$$u(s) = k_p \left(1 + \frac{1}{\tau_i s} \right) + T_d s e(s).$$

$$\frac{v(s)}{e(s)} = k_p \left(\left(1 + \frac{1}{\tau_i s} \right) + T_d s \right).$$



(iii) $\frac{d}{dt} \text{collisions}_\text{ext} + \text{losses}$

these losses due to scattering collisions at unit rate
and particles out of pattern

(iv) $\frac{d}{dt} \text{collisions}_\text{int} + \text{losses}$ (i)
losses due to interactions of particles (ii)

(v) $\frac{d}{dt} \text{collisions}_\text{loss} + \text{scattering} + \text{losses}$

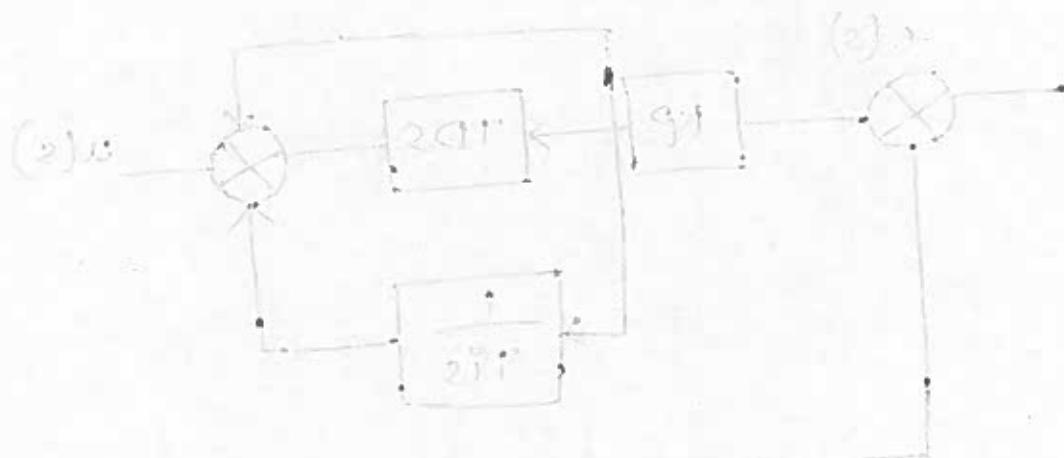
$$(i) \Rightarrow \frac{d}{dt} + (1) + (2) = (1)v$$

$$(i) \Rightarrow \frac{d}{dt} \text{scattering} + (1) = \frac{q_1}{q_1} + (1) \cdot q_1 = (1)v$$

$$(2) \Rightarrow 2 \cdot q_1 + \frac{(2)}{2} = \frac{q_1}{q_1} + (2) \cdot q_1 = (2)v$$

$$(2) \Rightarrow (2b)^2 + \left(\frac{1}{2} + 1\right)q_1 = (2)v$$

$$(2b)^2 + \left(\frac{1}{2} + 1\right)q_1 = \frac{(2)v}{(2)b}$$



Effect of PID controller.

15

The proportional controller stabilize the system but produces a steady state error.
'I' controller reduces the steady state error.
'D' controller reduces the rate of change of error.

collections 45

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2. Classical Control design techniques

17

All the control systems are designed to achieve specific objectives the certain requirements are designed for the control system.

- a) A good control system
- b) It has less errors
- c) Good accuracy
- d) Good stability
- e) Good speed response etc.

In practice, if a system is to be redesigned so has to meet the required specifications.

It is necessary to alter the system by adding an external device to it. Such as redesign (B) alteration of a system using a External suitable device is called a Compensation of Control Systems.

The compensator is provided whatever is missing in a system so has to achieve required performance.

Types of Compensation network:

There are three types of a Compensation networks.

They are:

(i) Lead network (B) Lead Compensation:

When a sinusoidal input is applied to a network at it produces a sinusoidal steady output having a phase

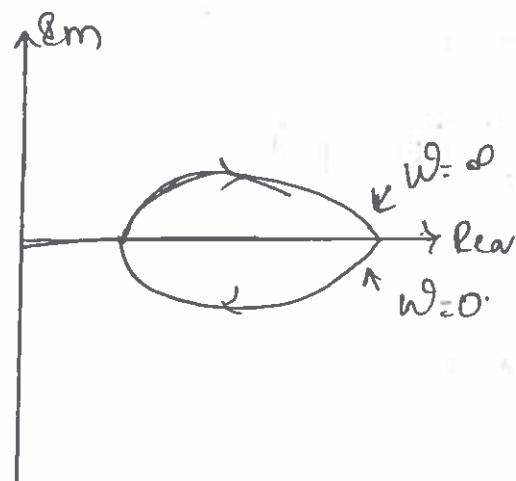
$$\phi = \tan^{-1}(\omega T_1) + \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_1/\beta) - \tan^{-1}(\beta T_2 \omega)$$

when $\omega = 0$

$$M=1, \phi=0$$

when $\omega = \infty$

$$M=1, \phi=0$$



State Space Analysis of Continuous System.

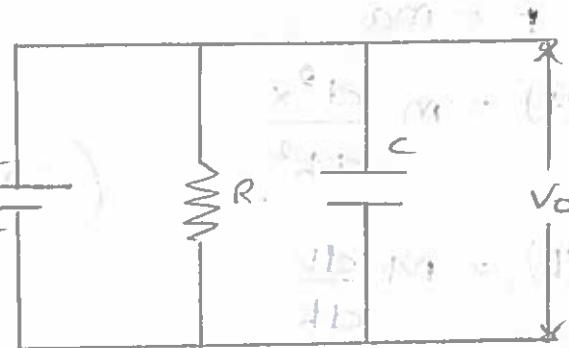
Disadvantages of Conventional methods :-

- * The method is insufficient and difficult to give complete time domain solution of higher order systems.
- * It is not very much convenient for the analysis of multiple input multiple output system.
- * It gives analysis of systems for some specific type of inputs like step, Ramp, Parabolic inputs etc.
- * It is only applicable to linear time invariant system.
- * The Bode plot, root locus etc. are basically trial and error methods which fails to give the optimal solution required.
- * It is not applicable to non-linear systems.
- * It is only applicable for zero initial conditions.
- * It cannot understand the matrix notation.

→ Advantages of State Variable Analysis:-

- * It is applicable for linear and non-linear Systems.
- * It is applicable for time variant and time invariant Systems.
- * This method takes into an account the effect of all initial conditions.
- * It can be conveniently applied to multiple input and multiple output system.
- * It is applicable to digital computers.
- * The vector matrix notation greatly simplifies the mathematical representation of the system.
- * Any type of input can be considered for designing the system.

Concept of State:



* Concept of total internal state of the system

Considering all initial conditions are called

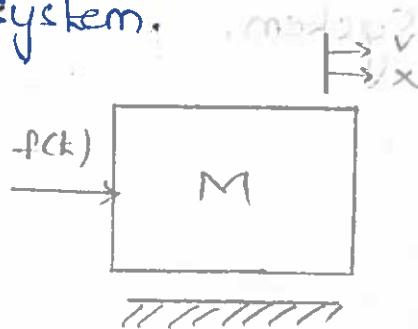
State variable analysis (or) state space analy-
sis.

* The output is not only depends on the input but also depends only on the initial conditions. are called the system in memory.

(or) Dynamic Systems. Whereas the capacitor is replaced by another resistor 'R₂' the output will depends on the only input this type of system is called system with zero memory.

(or) Static System.

Ex: Consider another example for simple mechanical system.



According to Newton's 2nd law motion.

$$F = ma$$

$$f(t) = m \frac{d^2x}{dt^2}$$

$$\left(\because v = \frac{dx}{dt} \right)$$

$$f(t) = m \frac{dv}{dt}$$

Integrate above equation on both sides

w.r.t 't'

$$v = \frac{1}{m} \int_{-\infty}^t f(t) dt$$

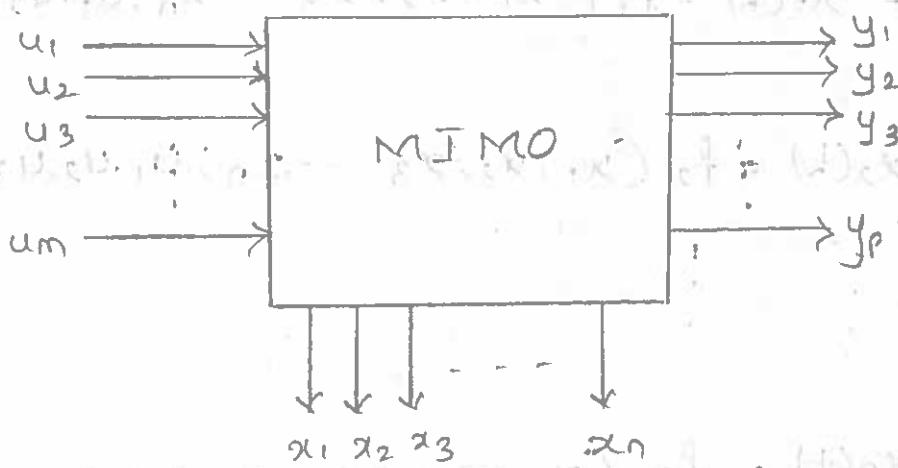
$$v = \frac{1}{m} \left[\int_{-\infty}^{t_0} f(t) dt + \int_{t_0}^t f(t) dt \right]$$

$$v = v_0(t_0) + \frac{1}{m} \int_{t_0}^t f(t) dt$$

for the same input $f(t)$ we can get the different values of velocities.

State model of a linear system:-

Consider multiple input and multiple output linear system.



Where $m = \text{number of inputs}$

$P = \text{number of outputs}$

$n = \text{number of state variables.}$

State :-

minimum set of variables are called

State.

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}_{m \times 1} \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_P(t) \end{bmatrix}_{P \times 1}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n \times 1}$$

Where $u(t)$, $y(t)$, $x(t)$ are the column

vectors of order $m \times 1$, $P \times 1$, $n \times 1$ respectively.

$$\frac{dx_1}{dt} = x_1(t) = f_1(x_1, x_2, x_3, \dots, x_n; u_1, u_2, u_3, \dots, u_m)$$

$$\frac{dx_2}{dt} = x_2(t) = f_2(x_1, x_2, x_3, \dots, x_n; u_1, u_2, u_3, \dots, u_m)$$

$$\begin{matrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{matrix}$$

$$\frac{dx_n}{dt} = x_n(t) = f_n(x_1, x_2, x_3, \dots, x_n; u_1, u_2, u_3, \dots, u_m)$$

for such a system the state variable representation can be arranged in the form of n 'th order differential equation.

where, $f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$ is a functional operator.

Integrating the above equation:

$$v(t) = v(t_0) + \int_{t_0}^t v(t) dt$$

$$v_i(t) = v_i(t_0) + \int_0^t f_i(x_1, x_2, x_3, \dots, x_n; u_1, u_2, u_3, \dots, u_m) dt$$

where $i = 1, 2, 3, \dots$ etc.

The functional equation can be expressed in terms of linear combination of system states and inputs as

$$\frac{dx_1}{dt} = x_1(t) = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_n x_n \\ + b_{11}u_1 + b_{12}u_2 + \dots + b_m u_m.$$

$$\frac{dx_2}{dt} = x_2(t) = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n \\ + b_{21}u_1 + b_{22}u_2 + \dots + b_{2m}u_m.$$

$$\frac{dx_n}{dt} = x_n(t) = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}u_1 \\ + b_{n2}u_2 + \dots + b_{nm}u_m.$$

The above equation can be written in matrix form as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$x(t) = Ax(t) + Bu(t)$$

is called state equation.

where $x(t)$ = State vector matrix.

$u(t)$ = Input vector matrix.

A = System matrix (or) Evaluation matrix.

B = Input matrix (or) Control matrix.

Similarly the output variables at a time 't'

can be expressed as the linear combinations of the input variables and state variables.

$$y_1(t) = C_{11}x_1 + C_{12}x_2 + \dots + C_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \dots + d_{1m}u_m$$

$$y_2(t) = C_{21}x_1 + C_{22}x_2 + \dots + C_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \dots + d_{2m}u_m$$

⋮
⋮
⋮

$$y_p(t) = C_{p1}x_1 + C_{p2}x_2 + \dots + C_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \dots + d_{pm}u_m$$

The above equation can be written in matrix form as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{p1} & C_{p2} & \dots & C_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ d_{21} & d_{22} & \dots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{p1} & d_{p2} & \dots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$y(t) = Cx(t) + Du(t)$$

is called output equation.

where : $y(t)$ = output vector matrix.

C = output matrix (or) observation

matrix.

D = Direct transmission matrix.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

is called state model of a linear system.

State Variable representation :-

The State Variables are minimum number of variables which are associated with all the initial conditions of the system as their sequence is not important.

* To obtain the state model for a given system

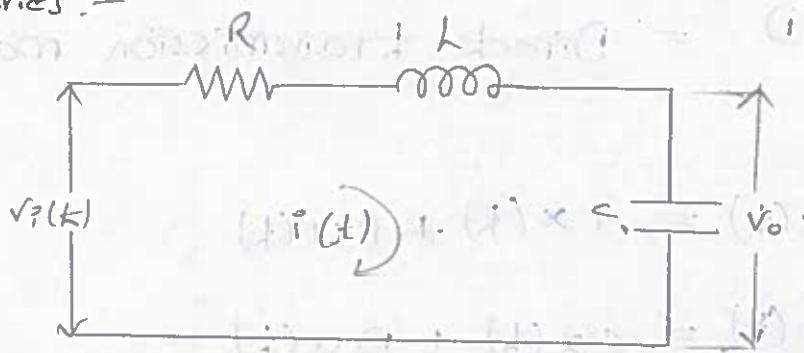
it is necessary to select the state variables.

* for the electrical system the currents through various inductors and voltages across the various capacitors are selected to be the state variables then by any method of network analysis.

* In the mechanical systems the displacements and velocities of energy storing elements such as spring and friction are selected as the state variables.

Obtain the state model of electrical system.

(RLC) Series:-



There are two state variables.

i.e., $x_1(t)$ and $x_2(t)$

$x_1(t) = i_L(t)$ = Current through the inductor.

$x_2(t) = V_C(t)$ = Voltage across the capacitor.

$$V(t) = V_i(t)$$

$$V_o(t) = y(t)$$

Applying KVL to the circuit.

$$V_i(t) = R_i(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int i(t) dt$$

$$V_o(t) = \frac{1}{C} \int i(t) dt$$

$$V_i(t) = R_i(t) + L \frac{di(t)}{dt} + V_o(t)$$

$$\frac{d}{dt} x_1(t) = \frac{1}{L} u(t) - \frac{R}{L} x_1(t) - \frac{1}{L} y(t)$$

$$x_1'(k) = \frac{1}{L} u(k) - \frac{R}{L} x_1(k) - \frac{1}{L} y(k)$$

$$x_1'(k) = \frac{1}{L} u(k) - \frac{R}{L} x_1(k) - \frac{1}{L} x_2(k) \quad \text{--- (1)}$$

$$V_o(k) = \frac{1}{C} \int i(k) dk$$

Diff w.r.t k

$$\frac{dV_o(k)}{dk} = \frac{1}{C} i(k) \quad \left(\begin{array}{l} \because x_2(k) = y(k) = V_o(k) \\ = v_c(k) \end{array} \right)$$

$$\frac{dx_2}{dt} = \frac{1}{C} x_1(k)$$

$$x_2'(k) = \frac{1}{C} x_1(k) \quad \text{--- (2)}$$

The above two equations can be written in matrix form as

$$x'(k) = Ax(k) + Bu(k)$$

$$\begin{bmatrix} x_1'(k) \\ x_2'(k) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(k)$$

The output equation is $y(k) = V_o(k)$

$$y(k) = x_2(k)$$

$$y(k) = [0 \ 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + [0] u(k)$$

$$y(k) = Cx(k) + Du(k)$$

The equations.

$$x'(k) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

are state model eqns.

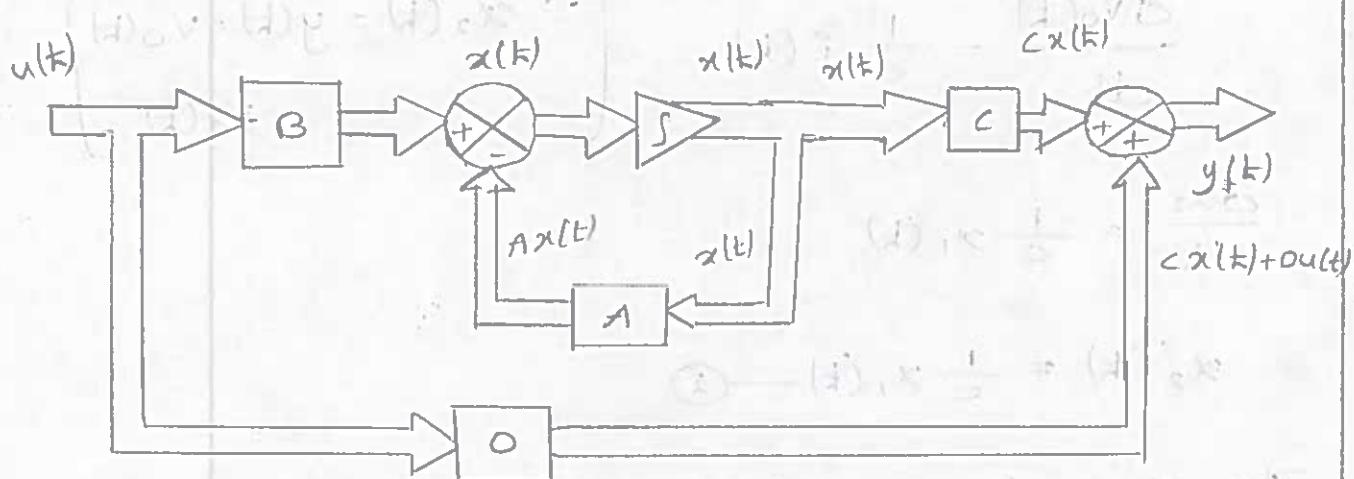
State diagram representation :-

The state equation is

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Output equation is

$$y(t) = cx(t) + du(t)$$

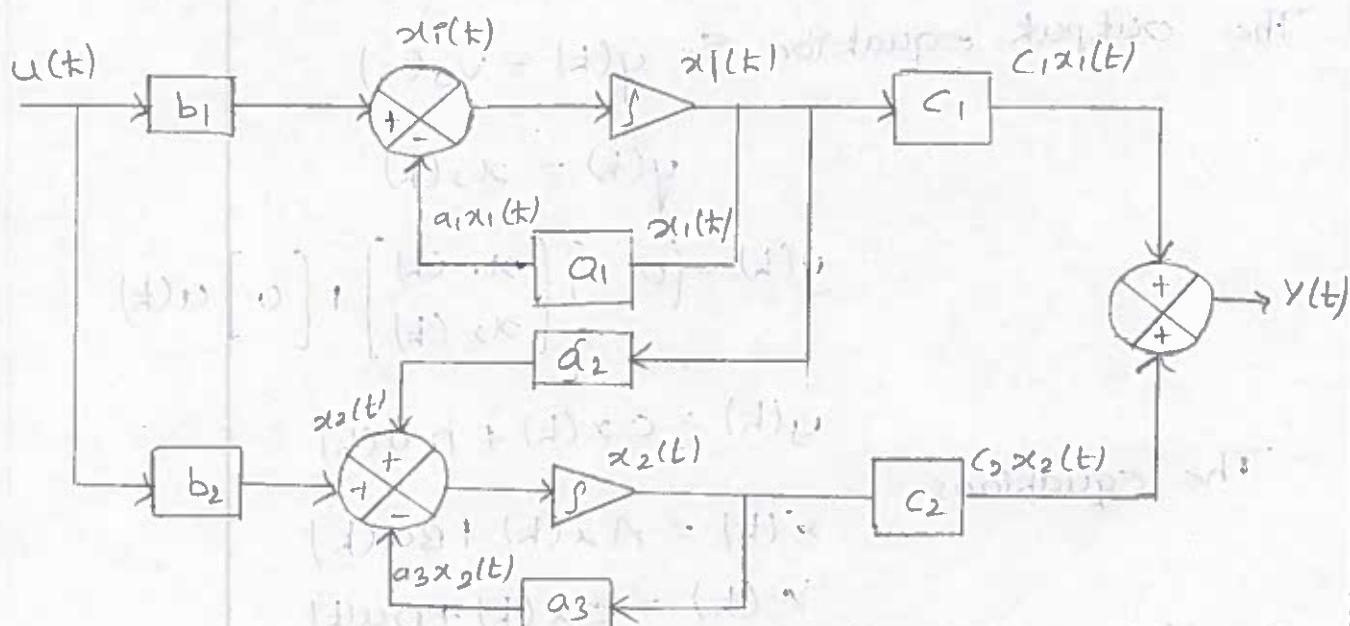


Obtain the state diagram of single input and single output system represented by the equations

$$\dot{x}_1(t) = a_1 x_1(t) + b_1 u(t)$$

$$\dot{x}_2(t) = a_2 x_1(t) + a_3 x_2(t) + b_2 u(t)$$

$$y(t) = c_1 x_1(t) + c_2 x_2(t)$$



7

State Space Representation using Phase Variables.

State model from differential equations:

Consider a linear Continuous time system represented by n^{th} order differential equation.

$$y(t)^n + a_{n-1} y(t)^{n-1} + a_{n-2} y(t)^{n-2} + \dots + a_1 y'(t) + a_0 = b_0 + b_1 u(t) + b_2 + \dots + b_m u(t)^m$$

$y^n(t) = \frac{dy^n(t)}{dy}$ = n^{th} order derivative of $y(t)$.

for time invariant system, the co-efficients

$a_{n-1}, a_{n-2}, \dots, a_0, b_0, b_1, \dots, b_m$ are constants for the system.

$y(t)$ = Output Variable.

$u(t)$ = Input Variable.

$y(0), y'(0), \dots, y^{n-1}$ represents the initial conditions of the system.

Consider the simple case of the system in which derivatives of control force $u(t)$ are absent. Thus

$$u(t) = u(t) = \dots = u^m(t) = 0$$

All the state equations are $x_1(t) = y(t)$

$$x_2(t) = x_1(t) = y'(t)$$

$$x_3(t) = x_2(t) = x_1(t) = y''(t)$$

$$x_n(t) =$$

$$\dot{x}_1(k) = x_2(k)$$

$$\dot{x}_2(k) = x_3(k)$$

$$\dot{x}_3(k) = x_4(k)$$

$$\dot{x}_{n-1}(k) = x_n(k)$$

$$\dot{x}_n(k) = x_{n+1}(k)$$

Note that only 'n' variables are to be designed

State model of $\dot{x}_n(k)$

$$y_n(k) + a_{n-1} y(k)^{n-1} + \dots + a_0 y(k) = b_0 u$$

$$x_n(k) + a_{n-1} x_n(k) + a_{n-2} x_{n-1}(k) + \dots + a_0 x_1(k) + a_{n-1} x_n(k) = b_0 u$$

$$\dot{x}_n(k) = -a_0 x(k) - a_1 x_2(k) - \dots - a_{n-2} x_{n-1}(k) - a_{n-1}$$

$$x_n(k) + b_0 u$$

The above equations can be written in matrix form as.

$$\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ \vdots \\ x_n(k) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -a_n & -a_{n-1} & a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b \end{bmatrix} u(k)$$

$$x(k) = Ax(k) + Bu(k)$$

Here the matrix A has a very special form. It has all ones in the upper off diagonal. Its last row is composed of the negative of the co-efficients of the original differential equation, and all other elements are zero. This form of matrix ' A ' is known as Bush form.

- * Also note that 'B' matrix has all its elements except the last element are zero. The output being $y = x_1$, the output equation is given by.

$$y(t) = [1 \ 0 \ 0 \ \dots \ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$y(t) = cx(t) + Du(t)$$

The advantage is using phase variables for.

State space modeling is that the system.

State model can be written directly by

Inspection from the differential equation

governing the system.

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Problem 2:

Construct the state model using phase variables if the system is described by the differential equation.

$$\frac{d^3y(t)}{dt^3} + u \frac{d^2y(t)}{dt^2} + 7 \frac{dy(t)}{dt} + 2y(t) = 5u(t)$$

Sol:

$$x_1(t) = y(t)$$

$$x_2(t) = x_1'(t) = y'(t) = \frac{dy(t)}{dt}$$

$$x_3(t) = x_2'(t) = x_1''(t) = y''(t) = \frac{d^2y(t)}{dt^2}$$

$$x_1'(t) = x_2(t) \quad \text{--- (1)}$$

$$x_2'(t) = x_3(t) \quad \text{--- (2)}$$

$$\frac{d^3y(t)}{dt^3} + \frac{d}{dt} \left(\frac{d^2y(t)}{dt^2} \right) = \frac{dx_3(t)}{dt} = x_3'(t)$$

$$x_3'(t) + 4x_3(t) + 7x_2(t) + 2x_1(t) = 5u(t)$$

$$x_3'(t) + 5u(t) - 4x_3(t) - 7x_2(t) - 2x_1(t) = 0 \quad \text{--- (3)}$$

The above three equations can be written in matrix form.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u(t)$$

$$\dot{x}^*(t) = Ax(t) + Bu(t)$$

The output equation is

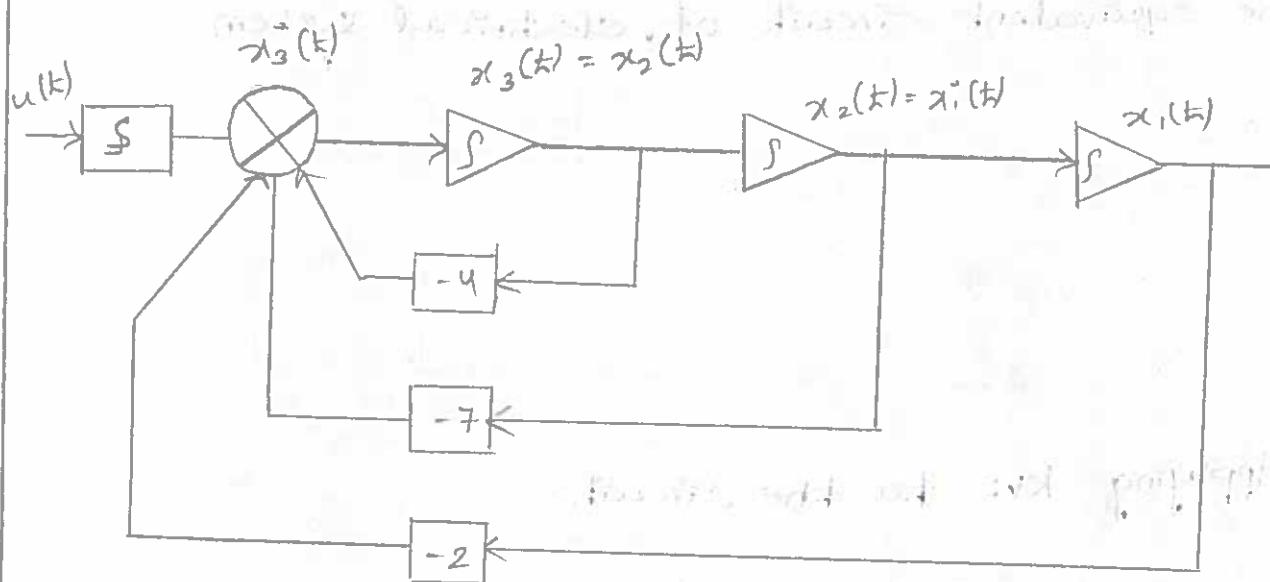
$$y(t) = x_1(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$

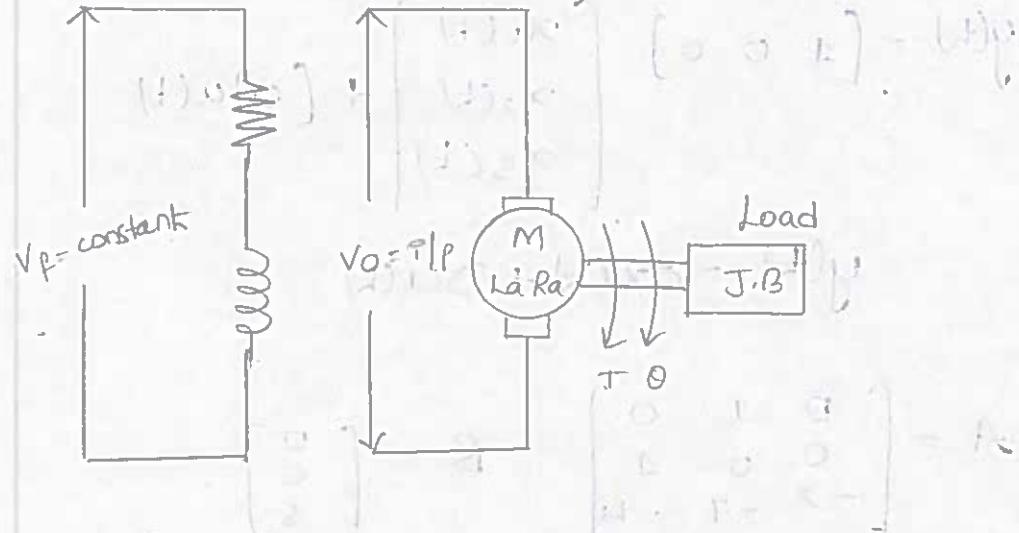
$$y(t) = Cx(t) + Du(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$C = [1 \ 0 \ 0] \quad D = [0]$$

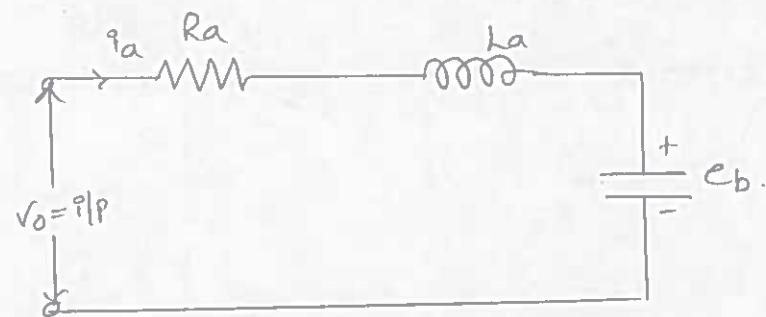


Obtain the State model of Armature Controlled DC motor.



Theory refer in 1st unit.

The equivalent circuit of electrical system



Applying KVL to the circuit.

$$V_o = R_a i_a + L_a \frac{di_a}{dt} + e_b \rightarrow \textcircled{1}$$

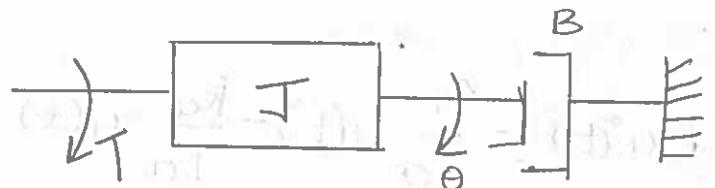
The torque of DC motor is directly proportional to armature current and flux in field winding.

$$T \propto i_a \phi$$

$$T \propto i_a$$

$$T = k_e i_a \rightarrow \textcircled{2}$$

The equivalent circuit of mechanical system.



from Newton's 2nd law.

$$T = J \frac{d\theta}{dt^2} + B \frac{d\theta}{dt} \rightarrow \textcircled{3}$$

The Back E.M.F of DC motor is proportional to angular velocity of the body.

$$e_b \propto \frac{d\theta}{dt}$$

$$e_b = k_b \frac{d\theta}{dt} \rightarrow \textcircled{4}$$

Substituting equation $\textcircled{4}$ in equation $\textcircled{1}$.

from $\textcircled{1}$

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + e_b$$

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + k_b \frac{d\theta}{dt}$$

$$\frac{di_a}{dt} = \frac{V_a}{L_a} - \frac{R_a}{L_a} i_a - \frac{k_b}{L_a} \omega \rightarrow \textcircled{5}$$

$$T = J \frac{d\omega}{dt} + B \omega \quad (\because \text{from } \textcircled{3}) \quad \left[\omega = \frac{d\theta}{dt} \right]$$

$$\frac{d\omega}{dt} = \frac{T}{J} - \frac{B}{J} \omega \quad [\text{from } \textcircled{2}]$$

$$\Rightarrow \frac{d\omega}{dt} = \frac{k_e i_a}{J} - \frac{B}{J} \omega \rightarrow \textcircled{6}$$

$$\frac{d\theta}{dt} = \omega \rightarrow \textcircled{7}$$

Choose the state variables as

$$I_a = x_1(t)$$

$$\omega = x_2(t)$$

$$\theta = x_3(t)$$

$$u(t) = V_a.$$

$$\frac{dx_1(t)}{dt} = \dot{x}_1(t) = \frac{1}{L_a} u(t) - \frac{R_a}{L_a} x_1(t) - \frac{k_b}{L_a} x_2(t) \rightarrow \textcircled{8}$$

$$\frac{dx_2(t)}{dt} = \dot{x}_2(t) = \frac{k_L}{J} x_1(t) - \frac{B}{J} x_2(t) \rightarrow \textcircled{9}$$

$$\frac{dx_3(t)}{dt} = \dot{x}_3(t) = x_2(t) \rightarrow \textcircled{10}$$

The above equation $\textcircled{8}$, $\textcircled{9}$, $\textcircled{10}$ can be

written in matrix form as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{k_b}{L_a} & 0 \\ \frac{k_L}{J} & -\frac{B}{J} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$\boxed{\dot{x}(t) = Ax(t) + Bu(t)}$$

This is state equation. $y_1 = \omega$. $y_2 = \theta$.

[Let the desired output be I_a , ω and θ .

Let us equate the desired output quantities to standard notation y_1 , y_2 and y_3 as shown below.

$$y_1 = I_a$$

$$y_2 = \omega$$

$$y_3 = \theta.$$

On relating the outputs to state variables we get

on relating the outputs to state variables.

We get

$$y_1 = x_2(t)$$

$$y_2 = x_3(t)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0] U(t).$$

$$y(t) = x(t) + DU(t)$$

Where $A = \begin{bmatrix} -R_F & 0 & 0 \\ L_F & -R/J & 0 \\ K_F \frac{1}{\sigma} & 0 & 0 \end{bmatrix}$

$$B = \begin{bmatrix} 1/U \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = [0]$$

* Derivation of Transfer Function of State model :-

The state model equations are

$$x(t) = Ax(t) + Bu(t)$$

$$y(t) = (x(t) + Du(t))$$

Consider the standard state model for linear time invariant system is shown in above equation.

On taking Laplace transform of above equation.

$$sX(s) - x(0) = Ax(s) + Bu(s)$$

$$sX(s) = Ax(s) + Bu(s) \rightarrow ①$$

$$y(s) = Cx(s) + Du(s) \rightarrow \textcircled{2} \quad (\because x(0) = 0)$$

From equation \textcircled{1}

$$\mathcal{L}x(s) - Ax(s) = Bu(s).$$

Multiplying with Identity matrix.

$$\boxed{I=1}$$

$$sI - A$$

$$sI - A \times (s) = Bu(s).$$

$$[sI - A]x(s) = Bu(s).$$

pre multiplying with $(sI - A)^{-1}$ on both sides of above

Equation \textcircled{1}

We get

$$[sI - A]^{-1} [sI - A]x(s) = [sI - A]^{-1} Bu(s).$$

$$x(s) = [sI - A]^{-1} Bu(s) \rightarrow \textcircled{3}$$

Substituting Eq \textcircled{3} in Eq \textcircled{2}.

$$y(s) = C[sI - A]^{-1} Bu(s) + Du(s)$$

$$y(s) = u(s) [C[sI - A]^{-1} B + D]$$

$$\frac{y(s)}{u(s)} = \left\{ C[sI - A]^{-1} B + D \right\}$$

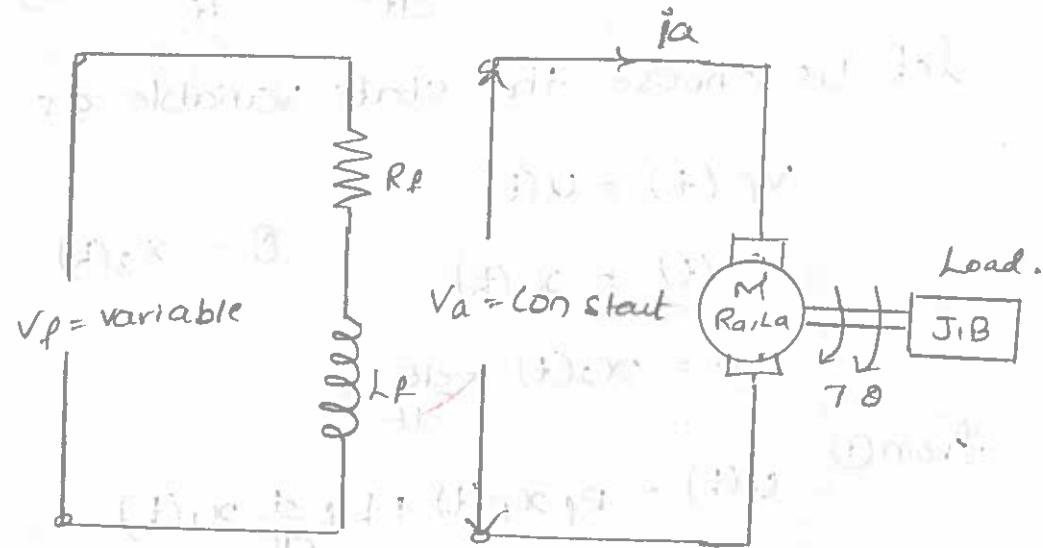
$$[sI - A]^{-1} = \frac{\text{Adj}[sI - A]}{|sI - A|} \left[\begin{array}{c} (sI - A) / (\text{det } sI - A) \\ \text{det } sI - A \end{array} \right]$$

$$(2) \times 1 + (2) \times 1 = (4) \times 1 \times 2$$

$$(2) \times 1 + (2) \times 1 = (2) \times 2$$

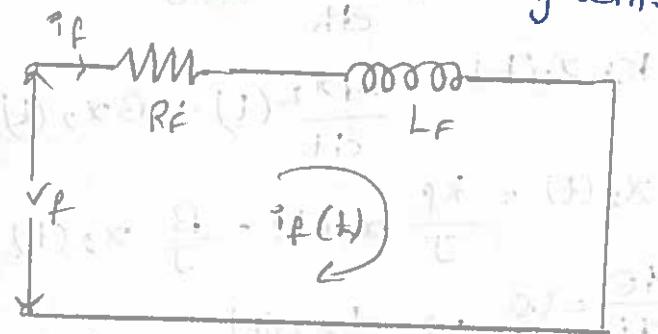
$$\text{unit-59 Pg-22/40}$$

Obtain the State model of field Controlled DC motor :-



As theory in 1st chapter.

Equivalent ckt of electrical system.



Applying KVL to the ckt

$$V_f = R_f i_f + L_f \frac{di_f}{dt} \quad \text{--- (1)}$$

The torque of dc motor is directly proportional to flux & current

$$T \propto i_f \phi \quad (\phi = \text{constant})$$

$$T = k_p i_f \quad \text{--- (2)}$$

Equivalent ckt of mechanical system is shown in below fig.



$$\text{Torque balance eqn} \quad T = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} \rightarrow \textcircled{3}$$

equating eqn $\textcircled{2}$ & $\textcircled{3}$

$$k_f i_f = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} \rightarrow \textcircled{4}$$

let us choose the state variable as

$$v_f(t) = u(t)$$

$$i_f(t) = x_1(t) \quad \theta = x_3(t)$$

$$\omega = x_2(t) = \frac{d\theta}{dt}$$

from $\textcircled{1}$

$$u(t) = R_f x_1(t) + L_f \frac{d}{dt} x_1(t)$$

$$\frac{d}{dt} x_1(t) = x_1(t) = \frac{1}{L_f} u(t) - \frac{R_f}{L_f} x_1(t) \rightarrow \textcircled{5}$$

from $\textcircled{4}$ $k_f x_1(t) = J \cdot \frac{d\omega}{dt} + B\omega$

$$k_f x_1(t) = \frac{J d x_2(t)}{dt} + B x_2(t)$$

$$\frac{d x_2(t)}{dt} = x_2(t) = \frac{k_f}{J} x_1(t) - \frac{B}{J} x_2(t) \rightarrow \textcircled{6}$$

$$\frac{d\theta}{dt} = \omega \Rightarrow \frac{d x_3(t)}{dt} = x_2(t)$$

$$x_3(t) = x_2(t) \rightarrow \textcircled{7}$$

The above eqns $\textcircled{5}$, $\textcircled{6}$, $\textcircled{7}$ is written in matrix form as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -R_f/L_f & 0 & 0 \\ k_f/J & -B/J & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1/L_f \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

let $y_1 = \omega$, $y_2 = \theta$ on relating the O/P to state variables, we

get $y_1 = x_2(t)$; $y_2 = x_3(t)$ can be written in matrix form as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = Cx(t) + Du(t)$$

where, $A = \begin{bmatrix} -R_f/L_f & 0 & 0 \\ k_f/J & -B/J & 0 \\ 0 & 1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1/L_f \\ 0 \\ 0 \end{bmatrix}$; $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $D = 0$

① Determine transfer matrix - for MI MO system.

is given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol The transfer function matrix is

$$T.F = C[S\mathbf{I} - A]^{-1}B + D$$

Given that

$$T.F = C[S\mathbf{I} - A]^{-1}B + D$$

$$A = \begin{bmatrix} 0 & 3 \\ 2 & -5 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, D = [0]$$

$$S\mathbf{I} - A = S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ 2 & -5 \end{bmatrix}$$

$$S\mathbf{I} - A = \begin{bmatrix} S & -3 \\ 2 & S+5 \end{bmatrix}$$

$$(S\mathbf{I} - A)^{-1} = \frac{\text{adj}[S\mathbf{I} - A]}{|S\mathbf{I} - A|}$$

$$\text{adj}[S\mathbf{I} - A] = \begin{bmatrix} S+5 & 3 \\ -2 & S \end{bmatrix}$$

$$\det(S\mathbf{I} - A) = S^2 + 5S + 6$$

$$(S\mathbf{I} - A)^{-1} = \frac{1}{S^2 + 5S + 6} \begin{bmatrix} S+5 & 3 \\ -2 & S \end{bmatrix}$$

$$T.F = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s+s & 3 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{s^2 + ss + 6}$$

$$T.F = \frac{1}{s^2 + ss + 6} \begin{bmatrix} 3s+14 & 3s+14 \\ s+8 & s+8 \end{bmatrix}$$

② The state equation of a linear time invariant system is given as $x' = \begin{bmatrix} 0 & s \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$ and

$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$ find the transfer function and draw the state diagram.

Soln Given:-

$$x' = \begin{bmatrix} 0 & s \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \text{ and } y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

$$A = \begin{bmatrix} 0 & s \\ -1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \end{bmatrix}$$

$$T.F = [(sI - A)^{-1} B + D]$$

$$sI - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & s \\ -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & s \\ -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} s & -s \\ 1 & s+2 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)}$$

$$\text{Adj}[sI - A] = \begin{bmatrix} s+2 & s \\ -1 & s \end{bmatrix}$$

$$\det(sI - A) = (s+2)s - (-1)s = s^2 + 2s + s$$

$$T.F [1, 1] \left[\begin{array}{cc} s+2 & s \\ -1 & s \end{array} \right] \frac{1}{s^2 + 2s + s} [1, 1] + [0]$$

$$\left[\begin{array}{cc} s+1 & s+s \\ -1 & s \end{array} \right] \frac{1}{s^2 + 2s + s} [1, 1] + [0]$$

$$\left[\begin{array}{cc} s+1 & s+s \\ -1 & s \end{array} \right] [1] \frac{1}{s^2 + 2s + s}$$

$$T.F = [2s+6] \frac{1}{s^2 + 2s + s}$$

$$T.F = \frac{Y(s)}{U(s)} = \frac{2s+6}{s^2 + 2s + s}$$

$$(s^2 + 2s + s) Y(s) = (2s+6) U(s).$$

$$s^2 Y(s) + 2s Y(s) + s Y(s) = (2s+6) U(s).$$

$$\frac{d^2 y(t)}{dt^2} + 2 \frac{dy(t)}{dt} + s y(t) = 2 \frac{d}{dt} U(t) + 6 U(t)$$

Let us choose the state variables as

$$x_1(t), x_2(t)$$

then

$$x_1(t) = y(t)$$

$$x_2(t) = x_1(t) = y(t)$$

The state equations are

$$x_1'(t) = x_2(t) \rightarrow ①$$

$$\frac{d}{dt} x_2(t) + 2x_2(t) + s x_1(t) = 2 \frac{d}{dt} U(t) + 6 U(t).$$

$$x_2'(t) = -2x_2(t) - sx_1(t) + 2 \frac{d}{dt} U(t) + 6 U(t).$$

③ Obtain the state model of the system whose transfer function is given by $\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1}$.

Sol: Given

Transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1}$$

$$s^3 y(s) + 4s^2 y(s) + 2s y(s) + y(s) = 10 U(s) \rightarrow ①.$$

State model:-

The state model equation is

$$x_1(t) = A x_1(t) + B U(t) \rightarrow \text{state equation.}$$

$$y(t) = C x_1(t) + D U(t) \rightarrow \text{state equation.}$$

On taking I.L.T of 1st equation.

$$\frac{d^3 y(t)}{dt^3} + 4 \frac{d^2 y(t)}{dt^2} + 2 \frac{dy(t)}{dt} + y(t) = 10 u(t).$$

The given system is third order differential equation
So it has three state variable i.e $x_1(t)$, $x_2(t)$ and $x_3(t)$

$$x_1(t) = y(t)$$

$$x_2(t) = x_1'(t) = y'(t)$$

$$x_3(t) = x_2'(t) = x_1''(t) = y''(t)$$

$$x_1'(t) = x_2(t) \rightarrow ②$$

$$x_2'(t) = x_3(t) \rightarrow ③.$$

From above equation can written as;

$$\frac{d}{dt} \left(\frac{d^2 y(t)}{dt^2} \right) + 4 \frac{d^2 y(t)}{dt^2} + 2 \frac{dy(t)}{dt} + y(t) = 10 u(t).$$

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$$\frac{d}{dt}(x_3(t)) + 4x_3(t) + 2x_2(t) + x_1(t) = 104(t),$$

$$\frac{d}{dt}x_3(t) = 104(t) - 4x_3(t) - 2x_2(t) - x_1(t).$$

$$\frac{d x_3(t)}{dt} - x_3(t) = 104(t) - 4x_3(t) - 2x_2(t) - x_1(t) \quad \rightarrow (4)$$

The above three equation can be written in matrix form.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u(t).$$

$$x(t) = Ax(t) + Bu(t).$$

$$y(t) = [1 \ 0 \ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0] u(t).$$

$$y(t) = Cx(t) + Du(t).$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$C = [1 \ 0 \ 0] \quad D = [0].$$

* State Transition Matrix:-

let us consider the state equation of the system is

$$x(t) = Ax(t) + Bu(t) \quad \text{--- (1)}$$

let us assume a homogenous state..

equation of the system is $x(t) = Ax(t) \quad \text{--- (2)}$

$$x(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \dots + b_k t^k + b_{k+1} t^{k+1}$$

Differentiating equation (3) w.r.t (B) we get $\hookrightarrow (4)$.

$$x'(t) = 0 + b_1 + 2b_2 t + 3b_3 t^2 + \dots + kb_k t^{k-1} + k+1 b_{k+1} t^k \hookrightarrow (4)$$

Substituting eqn (3) and (4) in eqn (2).

$$b_1 + 2b_2 t + 3b_3 t^2 + \dots + kb_k t^{k-1} + (k+1)b_{k+1} t^k = A [b_0 + b_1 t +$$

$$+ b_2 t^2 + b_3 t^3 + \dots + b_k t^k + b_{k+1} t^{k+1}]$$

$$b_1 + 2b_2 t + 3b_3 t^2 + \dots + kb_k t^{k-1} + (k+1)b_{k+1} t^k +$$

$$Ab_0 + Ab_1 t + Ab_2 t^2 + Ab_3 t^3 + \dots + Ab_k t^k +$$

$$+ Ab_{k+1} t^{k+1}.$$

Compare co-efficients on both sides.

Constants

$$b_1 = Ab_0,$$

t term :-

$$Ab_2 = Ab_1$$

$$b_2 = \frac{1}{2} Ab_1$$

$$b_2 = \frac{1}{2} A (Ab_0)$$

$$\boxed{b_2 = \frac{1}{2!} A^2 b_0}$$

t^2 term :-

$$Ab_3 = Ab_2$$

$$b_3 = \frac{1}{3} A b_2$$

$$b_3 = \frac{1}{3} A \left(\frac{1}{2} A^2 b_0 \right)$$

$$b_3 = \frac{1}{6} A^3 b_0$$

$$\boxed{b_3 = \frac{1}{3!} A^3 b_0}$$

3-term:

$$4b_4 = Ab_3.$$

$$b_4 = \frac{1}{4} A b_3$$

$$b_4 = \frac{1}{4} A \left(\frac{1}{6} A^3 b_0 \right).$$

$$b_4 = \frac{1}{24} A^4 b_0.$$

$$\boxed{b_k = \frac{1}{k!} A^k b_0.}$$

$$\boxed{b_{k+1} = \frac{1}{(k+1)!} A^{k+1} b_0.}$$

$$\boxed{b_{k+1} = \frac{1}{(k+1)!} A^{k+1} b_0}$$

Substituting above values in eqn (3).

$$x'(t) = b_0 \left[1 + \frac{b_1}{b_0} t + \frac{b_2}{b_0} t^2 + \frac{b_3}{b_0} t^3 + \dots + \frac{b_k}{b_0} t^k + \frac{b_{k+1}}{b_0} t^{k+1} \right]$$

$$x'(t) = b_0 \left[1 + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots + \frac{1}{k!} A^{k+1} t^{k+1} \right]$$

$$x(t) = b_0 e^{At}$$

$$[\because e^{ax} = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots + x^n]$$

$$x(t) = e^{At} b_0.$$

When $t = 0.$

$$x(0) = e^{A(0)} b_0.$$

$$\boxed{x(0) = b_0}$$

$$\text{then } \boxed{x(t) = e^{At} x(0).}$$

Where e^{At} is a matrix known as state transition matrix and it is denoted by

$$\boxed{\phi t = e^{At}}$$

$$E^{-1} \{ (EI - AJ^T) \} = \phi(t) = e^{At}$$

* properties of state transition matrix

(1) $\phi(0) = I$

Proof :-

$$\phi(t) = e^{At}$$

$$\phi(t) = I + \frac{1}{1!} At + \frac{1}{2!} (At)^2 + \frac{1}{3!} (At)^3 + \dots$$

When $t = 0$:

$$\phi(0) = I + \frac{1}{1!} A(0) + \frac{1}{2!} A(0)^2$$

$$\boxed{\phi(0) = I}$$

(2) $\phi^{-1}(t) = \phi(-t)$.

Proof :-

$$\phi(t) = e^{At}$$

Multiply e^{-At} on both sides.

$$\phi(t) e^{-At} = e^{At} e^{-At}$$

$$\phi(t) e^{-At} = e^{At} e^{-At}$$

$$\phi(t) \cdot e^{-At} = I$$

Multiply $\phi^{-1}(t)$ on both sides.

$$\phi(t) \phi^{-1}(t) e^{-At} = \phi^{-1}(t).$$

$$e^{-At} = \phi^{-1}(t).$$

$$\phi^{-1}(t) = e^{-At}$$

$$\boxed{\phi^{-1}(t) = \phi(-t)}$$

$$\begin{aligned} & \therefore \phi(t) = e^{At} \\ & \phi(-t) = e^{-At} \end{aligned}$$

Hence proved. unit-5, Pg - 32/40

$$\textcircled{3} \quad \phi(t_2 - t_1) \phi(t_1 - t_0) = \phi(t_2 - t_0)$$

Proof:-

W.K.T

$$\phi(t) = e^{At}$$

$$\text{then } \phi(t_2 - t_1) \phi(t_1 - t_0) = e^{A(t_2 - t_1)} e^{A(t_1 - t_0)} \\ = e^{At_2 - At_1 + At_1 - At_0}.$$

$$\text{. (15) } [e^{At_2} - e^{At_1} + e^{At_1} - e^{At_0}] = A$$

$$= e^{At_2 - At_0} \quad \begin{cases} \phi(t) = e^{At} \\ \phi(t_2 - t_0) = e^{A(t_2 - t_0)} \end{cases}$$

$$\boxed{\phi(t_2 - t_0) = \phi(t_2 - t_0)}.$$

$$\textcircled{4} \quad \phi(t_1 + t_2) = \phi(t_1) \phi(t_2).$$

Proof:-

$$\text{W.K.T } \phi(t) = e^{At}$$

$$\text{then } \phi(t_1 + t_2) = e^{A(t_1 + t_2)} \\ = e^{At_1} \cdot e^{At_2}$$

$$\boxed{\phi(t_1 + t_2) = \phi(t_1) \cdot \phi(t_2)}$$

$$\textcircled{5} \quad [\phi(t)]^k = \phi(kt).$$

Proof:-

We know that

$$\phi(t) = e^{At}$$

$$[\phi(t)]^k = [e^{At}]^k \\ = e^{A(kt)}$$

$$(a^m)^n = a^{mn}$$

$$\boxed{[\phi(t)]^k = \phi(kt)}$$

problem:

$$(st-i)\phi = (st-i)\phi \quad (st-i)\phi$$

- * Find the state transition matrix for the following matrix.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

* Consider the matrix A, complete $e^{At} \cdot 4A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Sol: $\phi(t) = e^{At}$

$$L^{-1}\{(sI - A)^{-1}\} = \phi(t).$$

$$e^{At} = L^{-1}\{(sI - A)^{-1}\}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$sI - A = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s-0 & 0-1 \\ 0+2 & s+3 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)}$$

$$\text{adj}(sI - A) = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\det(sI - A) = s^2 + 3s + 2$$

$$[sI - A]^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$L^{-1}\{sI - A\} = L^{-1}\left\{ \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \right\}$$

unit $s_1 = -3, s_2 = -2$

$$= L^{-1} \left\{ \begin{array}{l} \frac{s+3}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} \end{array} \right. \left. \begin{array}{l} \frac{1}{(s+1)(s+2)} \\ \frac{s}{(s+1)(s+2)} \end{array} \right\}$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{-A}{(s+1)} + \frac{B}{s+2}$$

$$s+3 = A(s+2) + B(s+1)$$

$$\text{put } s = -1.$$

$$-1+3 = A(1)$$

$$\boxed{2 = A}$$

$$-2+3 = B(-2+1)$$

$$1 = -B$$

$$\boxed{B = -1}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$= A(s+1) + B(s+2) \quad \text{put } s = -2$$

$$\text{put } s = -1.$$

$$1 = B$$

$$-A = 1$$

$$\boxed{A = -1}$$

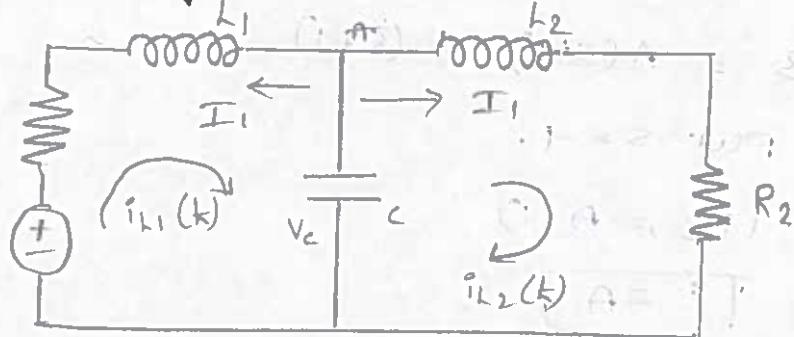
$$\frac{-2}{(s+1)(s+2)} = \boxed{A = -2} \quad \boxed{B = 2}$$

$$\frac{s}{(s+1)(s+2)} = \boxed{A = -1} \quad \boxed{B = 2}$$

$$= L^{-1} \left\{ \begin{array}{l} \frac{2}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} \end{array} \right. \left. \begin{array}{l} \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-1}{s+1} + \frac{1}{s+2} \end{array} \right\}$$

$$\Phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{2t} \end{bmatrix}$$

* Obtain the state model of Electrical Network shown in figure by choosing minimal no. of state variables.



SOLN

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Let us choose the state variables.

$$x_1(t), x_2(t) \text{ and } x_3(t)$$

$x_1(t) = i_{L_1}(t)$ = current through indicator L_1

$x_2(t) = i_{L_2}(t)$ = current through inductor L_2

$x_3(t) = v_c(t)$ = voltage across capacitor C

Here $u(t) = e(t)$, $y(t) = v(t)$.

Applying KVL to the top current $i_L(t)$:

$$-e(t) + R_1 i_{L_1}(t) + L_1 \frac{di_{L_1}(t)}{dt} + v_c = 0$$

$$-u(t) + R_1 x_1(t) + L_1 \frac{dx_1(t)}{dt} + x_3(t) = 0$$

$$\frac{dx_1(t)}{dt} = x_1(t) = \frac{1}{L_1} u(t) - \frac{R_1}{L_1} x_1(t) - \frac{1}{L_1} x_3(t) \quad \text{①}$$

Applying KVL to the loop current $i_{L_2}(t)$

$$L_2 \frac{di_{L_2}(t)}{dt} + R_2 i_{L_2}(t) - v_c = 0.$$

$$L_2 \frac{dx_2(t)}{dt} + R_2 x_2(t) - x_3(t) = 0.$$

$$\frac{dx_2(t)}{dt} = x_2(t) = \frac{1}{L_2} x_3(t) - \frac{R_2}{L_2} x_2(t) \rightarrow \textcircled{2}$$

$$\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 = 0.$$

$$i_{L_1}(t) + i_{L_2}(t) + C \frac{dv_c}{dt} = 0 \Rightarrow x_1(t) + x_2(t) + C \frac{dx_3}{dt} = 0.$$

$$x_3(t) = -\frac{1}{C} x_1(t) - \frac{1}{C} x_2(t) \rightarrow \textcircled{3}.$$

Above three eqns can be written in matrix form.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & -1/L_1 \\ 0 & -R_2/L_2 & 1/L_2 \\ -1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$x(t) = Ax(t) + Bu(t)$$

$$v_{o1}(t) = i_{L_2}(t) R_2$$

$$v_{o2}(t) = i_{L_1}(t) R_1$$

$$y_1(t) = i_{L_1}(t) R_2$$

$$y_2(t) = i_{L_2}(t) R_1$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & R_2 \\ R_1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0] u(t)$$

$$y(t) = Cy(t) + Du(t).$$

where

$$A = \begin{bmatrix} -R_1/L_1 & -1/L_1 \\ 0 & -\frac{R_2}{L_2} + \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} \end{bmatrix}; B = \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix}; C = \begin{bmatrix} 0 & R_2 \\ R_1 & 0 \end{bmatrix}$$

$$D = [0]$$

The State eqn and initial condition vector of an linear time invariant system are given below. Determine the soln of state eqn

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Given that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

we have

$$x'(t) = Ax(t) + Bx(t)$$

$$x^-(t) = e^{At}x(0)$$

$$e^{At} = L^{-1}\{(sI - A)^{-1}\}$$

$$(sI - A) = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{(sI - A)}$$

$$\text{adj}[sI - A] = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$|sI - A| = s^2 - s - s + 1 = s^2 - 2s + 1$$

$$(sI - A)^{-1} = \frac{\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}}{s^2 - 2s + 1}$$

$$L^{-1}\{(sI - A)^{-1}\} = L^{-1}\frac{1}{s^2 - 2s + 1} \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$= L^{-1} \begin{bmatrix} \frac{s-1}{(s-1)(s-1)} & 0 \\ \frac{1}{(s-1)^2} & \frac{s-1}{(s-1)(s-1)} \end{bmatrix}$$

$$= L^{-1} \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$L^{-1}(sI - A)^{-1} = \begin{bmatrix} e^{+t} & 0 \\ te^{+t} & e^t \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^t + 0 \\ t e^t + 0 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^t \\ t e^t \end{bmatrix}$$